Estimating the eigenvalues on Quaternionic Kähler Manifolds

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Abstract

We study geometric first order differential operators on quaternionic Kähler manifolds. Their principal symbols are related to the enveloping algebra and Casimir elements for Sp(1)Sp(n). This observation leads to anti-symmetry of the principal symbols and Bochner-Weitzenböck formulas for operators. As an application, we estimate their first eigenvalues.
Keywords: quaternionic Kähler manifolds, Bochner-Weitzenböck formulas, Casimir elements
2000MSC: 58J60. 17B35, 53C26,

1 Introduction

In differential geometry, Bochner-Weitzenböck formulas play an important role to provide vanishing theorems and eigenvalue estimates for geometric differential operators. The strategy of giving such formulas is to find out algebraic structure among symbols of operators. As an example, we consider the Dirac operator on a spin manifold,

$$D = \sum_i e_i \nabla_{e_i},$$

where ∇ is a covariant derivative and $\{e_i\}_i$ is a local orthonormal frame. The principal symbol of D is the Clifford multiplication, which satisfies the Clifford relation $e_i e_j + e_j e_i = -2\delta_{i,j}$. We rewrite this relation as

$$E_{ij} := e_i e_j + \delta_{i,j} = -(e_j e_i + \delta_{j,i}) = -E_{ji}. \tag{1.1}$$

On the other hand, setting $\nabla^2_{X,Y} := \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$, we know that the second order operator $\nabla^2_{e_i,e_j}$ satisfies a symmetric relation,

$$\nabla^2_{e_i,e_j} - R(e_i, e_j)/2 = \nabla^2_{e_j,e_i} - R(e_j, e_i)/2,$$

where R is the curvature of ∇. Combining this symmetry with the anti-symmetry (1.1), we have

$$\begin{aligned}(D^2 - \nabla^*\nabla - \kappa/4) &= \sum_{i,j} E_{ij}(\nabla^2_{e_i,e_j} - R(e_i,e_j)/2) \\ &= \sum_{i,j} -E_{ji}(\nabla^2_{e_j,e_i} - R(e_j,e_i)/2) = -(D^2 - \nabla^*\nabla - \kappa/4),\end{aligned}$$

and hence, $D^2 = \nabla^*\nabla + \kappa/4$. Thus the essential point is to search anti-symmetry for the principal symbols of operators.

As mentioned in [8], the principal symbols of first order geometric operators called *gradients* are controlled by the enveloping algebras of the Lie algebra $\mathfrak{so}(n)$ in a sense. From this observation, we have anti-symmetry such as (1.1) of the principal symbols. Then we can give all Bochner-Weitzenböck formulas for gradients on Riemannian or spin manifolds from a point of view of representation theory. In [7], gradients on Kähler manifolds are also discussed. Working on the enveloping algebra associated to the holonomy group $\mathrm{U}(n)$, we can produce Bochner-Weitzenböck formulas on Kähler manifolds.

In this paper, we discuss gradients and their Bochner-Weitzenböck formulas on quaternionic Kähler manifolds with holonomy group in $\mathrm{Sp}(1)\mathrm{Sp}(n)$. The gradient is a first order differential operator defined to be an irreducible component of covariant derivative on an associated vector bundle. Most of differential operators in quaternionic Kähler geometry are realized as gradients or twisted gradients.

In [16], U. Semmelmann and G. Weingart presented an excellent method of giving vanishing theorems and eigenvalue estimates for the Laplace operators and the Dirac operators on quaternionic Kähler manifolds. Their method is to consider twisted Dirac operators and compare their squares with the Laplace operators. On the other hand, our method is more direct and universal. We give Bochner-Weitzenböck formulas for gradients on any irreducible bundle concretely. Hence we can produce a lot of vanishing theorems and eigenvalue estimates. Thus our formulas are useful in various scenes of quaternionic geometry. As examples, we obtain lower bounds of the eigenvalues of the Laplace operators on differential forms. Our estimates not only cover the ones in [16] but are better.

This paper is organized as follows. In Section 2, we give some results on the enveloping algebra of $\mathfrak{sp}(n)$. Proposition 2.1 leads to anti-symmetry of symbols of gradients on quaternionic Kähler manifolds. In Section 3, we calculate the eigenvalues of Casimir elements and state notation used in this paper. In Section 4, we give anti-symmetry of the symbols as mentioned above. In Section 5, we define gradients on quaternionic Kähler manifolds and show their conformal covariance, and in Section 6 we give the first main theorem, Bochner-Weitzenböck formulas for gradients. After we discuss curvature endomorphisms on differential forms in Section 7, we give the second main theorem, eigenvalue estimates of the Laplace operators, in the last section. Furthermore, we obtain some vanishing theorems, Proposition 8.3 and Corollary 8.6.

Throughout this paper, we assume that the real dimension of a quaternionic Kähler manifold is greater than or equal to 8. For the 4-dimensional case, see [8].

2 Enveloping algebra and Casimir elements

Let E be a $2n$-dimensional complex vector space equipped with a complex symplectic structure σ_E, a quaternionic structure J_E, and a positive definite Hermitian inner product $\sigma_E(\cdot, J_E(\cdot))$. We fix a symplectic unitary basis

$$\{\epsilon_\alpha | \alpha = -n, -(n-1), \dots, -1, 1, \dots, n\}$$

such that

$$\sigma_E(\epsilon_\alpha, \epsilon_\beta) = \mathrm{sign}(\alpha)\delta_{\alpha,-\beta}, \quad J_E(\epsilon_\alpha) = \mathrm{sign}(\alpha)\epsilon_{-\alpha},$$

where $\mathrm{sign}(\alpha) = \pm 1$ is the sign of α.

The complex symplectic group $\mathrm{Sp}(n, \mathbb{C})$ on E is the group of automorphisms preserving σ_E and the symplectic group $\mathrm{Sp}(n)$ is the real subgroup of $\mathrm{Sp}(n, \mathbb{C})$ compatible with J_E. The second symmetric tensor product space $S^2(E)$ is isomorphic to the Lie algebra $\mathfrak{sp}(n, \mathbb{C})$ of $\mathrm{Sp}(n, \mathbb{C})$ by associating $\epsilon \odot \epsilon'$ in $S^2(E)$ with an endomorphism

$$(\epsilon \odot \epsilon')(u) := \sigma_E(\epsilon, u)\epsilon' + \sigma_E(\epsilon', u)\epsilon \quad \text{for each } u \text{ in } E.$$

In particular, we can choose $\{\epsilon_\alpha \odot \epsilon_\beta | \alpha + \beta \geq 0\}$ as a basis of $\mathfrak{sp}(n, \mathbb{C})$. But, to construct Casimir elements, it is better to employ $x_{\alpha\beta} := -\mathrm{sign}(\beta)\epsilon_\alpha \odot \epsilon_{-\beta}$ instead of $\epsilon_\alpha \odot \epsilon_\beta$ as a basis. This $x_{\alpha\beta}$ acts on E by

$$x_{\alpha\beta}(\epsilon_\nu) = \delta_{\beta,\nu}\epsilon_\alpha - \mathrm{sign}(\alpha\beta)\delta_{-\alpha,\nu}\epsilon_{-\beta},$$

and satisfies

$$x_{\alpha\beta} = -\mathrm{sign}(\alpha\beta)x_{-\beta-\alpha},$$
$$[x_{\alpha\beta}, x_{\mu\nu}] = \delta_{\beta,\mu}x_{\alpha\nu} - \delta_{\alpha,\nu}x_{\mu\beta} + \mathrm{sign}(\alpha\beta)(\delta_{-\beta,\nu}x_{\mu-\alpha} - \delta_{-\alpha,\mu}x_{-\beta\nu})$$

for $\alpha, \beta, \mu, \nu = \pm 1, \dots, \pm n$.

Let $U(\mathfrak{sp}(n,\mathbb{C}))$ be the universal enveloping algebra of $\mathfrak{sp}(n,\mathbb{C})$. The center $\mathfrak{Z}$ of $U(\mathfrak{sp}(n,\mathbb{C}))$ is characterized as the invariant sub-algebra of $U(\mathfrak{sp}(n,\mathbb{C}))$ under the adjoint action of $\mathrm{Sp}(n,\mathbb{C})$, whose elements are called *Casimir elements*. It is well-known how to construct Casimir elements generating $\mathfrak{Z}$ [14], [19]. For each nonnegative integer q, we define an element $x^q_{\alpha\beta}$ in $U(\mathfrak{sp}(n,\mathbb{C}))$ by

$$x^q_{\alpha\beta} := \begin{cases} \sum_{\alpha_1,\alpha_2,\dots,\alpha_{q-1}=\pm 1,\dots,\pm n} x_{\alpha\alpha_1}x_{\alpha_1\alpha_2}\cdots x_{\alpha_{q-1}\beta} & q \geq 1 \\ \delta_{\alpha,\beta} & q = 0. \end{cases}$$

Then the trace $c_q := \sum_\alpha x^q_{\alpha\alpha}$ is a Casimir element and the center $\mathfrak{Z}$ is generated by $c_2, c_4, \dots, c_{2n}$. We will need their translated elements in later sections. The translated elements are defined by

$$\hat{x}_{\alpha\beta} := x_{\alpha\beta} - (n + 1/2)\delta_{\alpha,\beta},$$

$$\hat{x}^q_{\alpha\beta} := \begin{cases} \sum_{\alpha_1,\alpha_2,\dots,\alpha_{q-1}=\pm 1,\dots,\pm n} \hat{x}_{\alpha\alpha_1}\hat{x}_{\alpha_1\alpha_2}\cdots \hat{x}_{\alpha_{q-1}\beta} & q \geq 1 \\ \delta_{\alpha,\beta} & q = 0, \end{cases}$$

$$\hat{c}_q := \sum_\alpha \hat{x}^q_{\alpha\alpha}.$$

In [6], the author showed the following.

Proposition 2.1. *Translated elements $\{\hat{x}^q_{\alpha\beta}\}_{q,\alpha,\beta}$ satisfy*

$$\hat{x}^q_{\alpha\beta} = \mathrm{sign}(\alpha\beta)\left\{(-1)^q\hat{x}^q_{-\beta-\alpha} - \frac{1-(-1)^q}{2}\hat{x}^{q-1}_{-\beta-\alpha} - \sum_{p=0}^{q-1}(-1)^p\hat{c}_{q-1-p}\hat{x}^p_{-\beta-\alpha}\right\}, \tag{2.1}$$

and

$$\hat{x}^q_{\alpha\beta} = \sum_{p=0}^{q}\binom{q}{p}\left(-n-\frac{1}{2}\right)^{q-p} x^p_{\alpha\beta}. \tag{2.2}$$

The translated Casimir elements $\{\hat{c}_q\}_q$ satisfy

$$2\hat{c}_{2q+1} = -\hat{c}_{2q} - \sum_{p=0}^{2q}(-1)^p\hat{c}_{2q-p}\hat{c}_p. \tag{2.3}$$

3 Representation of $\mathrm{Sp}(n)$ and eigenvalues of Casimir elements

We set $\mathfrak{h} := \mathrm{span}_{\mathbb{R}}\{\sqrt{-1}x_{ii} | i = 1,\dots,n\}$ as a maximal abelian subalgebra of $\mathfrak{sp}(n)$. We consider a finite-dimensional irreducible unitary $\mathrm{Sp}(n)$-module

V and decompose it into simultaneous eigenspaces with respect to $\mathfrak{h}$, $V = \bigoplus_\lambda V(\lambda)$. Each eigenvalue $\lambda = (\lambda^1, \dots, \lambda^n)$ is called weight, and a weight vector ϕ_λ in $V(\lambda)$ satisfies $x_{ii}\phi_\lambda = \lambda^i \phi_\lambda$ for $i = 1, \dots, n$. Ordering these weights lexicographically, we have the highest weight ρ which satisfies the dominant integral condition,

$$\rho = (\rho^1, \dots, \rho^n) \in \mathbb{Z}^n \text{ and } \rho^1 \geq \rho^2 \geq \cdots \geq \rho^n \geq 0.$$

Conversely, for a dominant integral weight ρ, we can construct an irreducible unitary $\mathrm{Sp}(n)$-module with highest weight ρ. Therefore we denote by (π_ρ, V_ρ) an irreducible unitary representation of $\mathrm{Sp}(n)$ and its infinitesimal one with highest weight ρ. When writing a weight, we denote by k_a a string of k with length a and sometimes omit a string of 0. For example, $(1_a) = (1_a, 0_{n-a})$ is a weight such that the first a components are 1 and the others are 0.

Example 3.1. The quaternionic vector space E is an irreducible $\mathrm{Sp}(n)$-module with highest weight (1_1). The second symmetric tensor product space $S^2(E) \simeq \mathfrak{sp}(n, \mathbb{C})$ has the highest weight (2_1). The exterior tensor product space $\Lambda^a(E)$ for $2 \leq a \leq n$ is not irreducible. Actually, by using the complex symplectic form σ_E, we can decompose $\Lambda^a(E)$ into irreducible components,

$$\Lambda^a(E) = \bigoplus_{p=0}^{[a/2]} \sigma_E^p \Lambda_0^{a-2p}(E).$$

Here, $\Lambda_0^a(E)$ is the so-called primitive space of $\Lambda^a(E)$, which is an irreducible $\mathrm{Sp}(n)$-module with highest weight (1_a).

Example 3.2. In quaternionic Kähler geometry, we often discuss an $\mathrm{Sp}(n)$-module with highest weight $(2_b, 1_{a-b})$ for $0 \leq b \leq a \leq n$. The representation space is realized as the top irreducible summand of $\Lambda_0^a(E) \otimes \Lambda_0^b(E)$. We denote it by $\Lambda_0^{a,b}(E)$.

Remark 3.1. By quaternionic structure J_E on E, we can put a quaternionic or real structure on each irreducible $\mathrm{Sp}(n)$-module [6]. In fact, there is a quaternionic (resp. real) structure on V_ρ in the case that $\sum_i \rho^i$ is odd (resp. even).

We shall calculate eigenvalues of Casimir element c_q on irreducible $\mathrm{Sp}(n)$-modules. We consider a tensor product space $V_\rho \otimes E$. The highest weights of irreducible summands in $V_\rho \otimes E$ are

$$\{\rho + \mu_\nu \mid \rho + \mu_\nu \text{ is dominant integral}, \nu = \pm 1, \dots, \pm n\},$$

where

$$\mu_\nu = \begin{cases} \mu_\nu := (\underbrace{0, \dots, 0}_{\nu-1}, 1, \underbrace{0, \dots, 0}_{n-\nu}) & \text{for } 1 \leq \nu \leq n, \\ \mu_\nu := -\mu_{-\nu} & \text{for } -n \leq \nu \leq -1. \end{cases}$$

Setting $V_{\rho+\mu_\nu} := \{0\}$ for $\rho+\mu_\nu$ without dominant integral condition, we can describe the irreducible decomposition of $V_\rho \otimes E$ as

$$V_\rho \otimes E = \bigoplus_{\nu=\pm 1,\ldots,\pm n} V_{\rho+\mu_\nu} = \bigoplus_{i=1,\ldots,n} (V_{\rho+\mu_i} \oplus V_{\rho-\mu_i}).$$

Each component $V_{\rho+\mu_\nu}$ is equipped with a Hermitian inner product of the restriction of the one on $V_\rho \otimes E$.

Let Π_ν be the orthogonal projection from $V_\rho \otimes E$ onto $V_{\rho+\mu_\nu}$. We define a linear mapping $p_\nu(\epsilon) : V_\rho \to V_{\rho+\mu_\nu}$ for each ϵ in E by

$$V_\rho \ni \phi \mapsto p_\nu(\epsilon)\phi := \Pi_\nu(\phi \otimes \epsilon) \in V_{\rho+\mu_\nu}, \tag{3.1}$$

and denote by $p_\nu(\epsilon)^*$ the adjoint map of $p_\nu(\epsilon)$ with respect to Hermitian inner products of V_ρ and $V_{\rho+\mu_\nu}$. To connect with these linear mappings and the enveloping algebra, we assign a constant w_ν called *the conformal weight* to $\rho+\mu_\nu$,

$$\begin{cases} w_i := -(\rho^i - i + 1) & \text{to } \rho+\mu_i \text{ for } i = 1,\ldots,n, \\ w_{-i} := \rho^i - i + 2n + 1 & \text{to } \rho+\mu_{-i} \text{ for } i = 1,\ldots,n, \end{cases}$$

and define the translated conformal weight $\hat{w}_\nu$ by

$$\hat{w}_\nu := w_\nu - (n + 1/2).$$

Then we have

Proposition 3.1 ([6],[14]).

$$\sum_{\nu=\pm 1,\ldots,\pm n} w_\nu^q p_\nu(\epsilon_\alpha)^* p_\nu(\epsilon_\beta) = \operatorname{sign}(\alpha\beta)\pi_\rho(x^q_{-\alpha-\beta}), \tag{3.2}$$

$$\sum_{\nu=\pm 1,\ldots,\pm n} \hat{w}_\nu^q p_\nu(\epsilon_\alpha)^* p_\nu(\epsilon_\beta) = \operatorname{sign}(\alpha\beta)\pi_\rho(\hat{x}^q_{-\alpha-\beta}), \tag{3.3}$$

$$\sum_{\alpha=\pm 1,\ldots,\pm n} p_\nu(\epsilon_\alpha)^* p_\nu(\epsilon_\alpha) = \frac{\dim V_{\rho+\mu_\nu}}{\dim V_\rho},$$

$$\pi_\rho(c_q) = \sum_{\nu=\pm 1,\ldots,\pm n} w_\nu^q \frac{\dim V_{\rho+\mu_\nu}}{\dim V_\rho}, \quad \pi_\rho(\hat{c}_q) = \sum_{\nu=\pm 1,\ldots,\pm n} \hat{w}_\nu^q \frac{\dim V_{\rho+\mu_\nu}}{\dim V_\rho}. \tag{3.4}$$

Without this proposition, we can compute the eigenvalues of c_q and $\hat{c}_q$ for $q = 0, 1, 2$ [14], [19]. On an irreducible $\mathrm{Sp}(n)$-module V_ρ,

$$\begin{aligned} \pi_\rho(c_0) = 2n, \quad \pi_\rho(c_1) = 0, \quad \pi_\rho(c_2) = 2\sum_{i=1}^n \rho^i \left(\rho^i + 2(n-i+1)\right), \\ \pi_\rho(\hat{c}_0) = 2n, \quad \pi_\rho(\hat{c}_1) = -2n^2 - n, \quad \pi_\rho(\hat{c}_2) = \pi_\rho(c_2) + 2n(n+1/2)^2. \end{aligned} \tag{3.5}$$

From (2.3), we also have the eigenvalue of c_3 and $\hat{c}_3$,

$$\pi_\rho(c_3) = (n+1)\pi_\rho(c_2), \quad \pi_\rho(\hat{c}_3) = -(2n+1/2)\pi_\rho(c_2) - 2n(n+1/2)^3. \tag{3.6}$$

To calculate the eigenvalues of higher Casimir elements, we need Proposition 3.1. In fact, there are formulas for the eigenvalues of higher Casimir elements [14], [19]. But those formulas are complicated to compute explicitly. On the other hand, D. Calderbank, P. Gauduchon and M. Herzlich gave a nice formula to calculate the eigenvalues of Casimir elements for the special orthogonal group SO(n) [3]. From a similar discussion, we have a formula of $\pi_\rho(c_q)$ calculated easily.

Proposition 3.2. *We denote by $\mathcal{N}$ the number of irreducible summands in $V_\rho \otimes E$. Then the relative dimension* $\dim V_{\rho+\mu_\nu}/\dim V_\rho$ *is given by*

$$\frac{\dim V_{\rho+\mu_\nu}}{\dim V_\rho} = -2(\hat{w}_\nu - (-1)^{\mathcal{N}}) \prod_{\substack{\nu' \neq \nu, \\ \rho+\mu_{\nu'} \text{ is dominant}}} \frac{\hat{w}_\nu + \hat{w}_{\nu'}}{\hat{w}_\nu - \hat{w}_{\nu'}}.$$

Of course, the relative dimension $\dim V_{\rho+\mu_\nu}/\dim V_\rho$ *is zero for $\rho+\mu_\nu$ without dominant integral condition. By the above equation and* (3.4)*, we can easily calculate the eigenvalues of c_q.*

Remark 3.2. The number $\mathcal{N}$ is odd if and only if the nth component ρ^n of the highest weight ρ is zero.

Example 3.3. We shall calculate eigenvalues of c_2 and c_4 on $V_{(2_b,1_{a-b})} = \Lambda_0^{a,b}(E)$. The Sp($n$)-module $V_{(2_b,1_{a-b})} \otimes E$ splits as

$$\begin{aligned} V_\rho \otimes E =& V_{\rho+\mu_1} \oplus V_{\rho+\mu_{b+1}} \oplus V_{\rho+\mu_{a+1}} \oplus V_{\rho+\mu_{-b}} \oplus V_{\rho+\mu_{-a}} \\ =& V_{(3,2_{b-1},1_{a-b})} \oplus V_{(2_{b+1},1_{a-b-1})} \oplus V_{(2_b,1_{a-b+1})} \oplus V_{(2_{b-1},1_{a-b+1})} \oplus V_{(2_b,1_{a-b-1})}. \end{aligned}$$

The next table of the relative dimension $\dim V_{\rho+\mu_\nu}/\dim V_\rho$ follows from Proposition 3.2.

Then we have

$$\begin{aligned} \pi_{(2_b,1_{a-b})}(c_2) =& 2a(2n-a+2) + 2b(2n-b+4), \\ \pi_{(2_b,1_{a-b})}(c_4) =& 2a(2n-a+2)(2n+3)(n+1) - 2a^2(2n-a+2)^2 \\ &+ 2b(2n-b+4)(2n+3)(n+3) - 2b^2(2n-b+4)^2. \end{aligned}$$

In the next section, we will discuss symbols of gradients on a quaternionic Kähler manifold whose holonomy group is in Sp(1)Sp(n). Then we shall state some facts for the Sp(1)-case. We consider a 2-dimensional complex vector space H with quaternionic structure J_H and symplectic structure σ_H. The

Table 1:

$\rho+\mu_\nu$	w_ν	relative dimension
$\rho+\mu_1$	-2	$\dfrac{2b(a+1)(2n-a+3)(2n-b+4)(n+2)}{(a+2)(b+1)(2n-a+4)(2n-b+5)}$
$\rho+\mu_{b+1}$	$b-1$	$\dfrac{(a-b)(2n-b+4)(2n-a-b+2)(n-b+1)}{(b+1)(a-b+1)(2n-a-b+3)(n-b+2)}$
$\rho+\mu_{a+1}$	a	$\dfrac{(a-b+2)(2n-a+3)(2n-a-b+2)(n-a)}{(a+2)(a-b+1)(2n-a-b+3)(n-a+1)}$
$\rho+\mu_{-b}$	$2n-b+3$	$\dfrac{b(a-b+2)(2n-a-b+4)(n-b+3)}{(a-b+1)(2n-b+5)(2n-a-b+3)(n-b+2)}$
$\rho+\mu_{-a}$	$2n-a+2$	$\dfrac{(a+1)(a-b)(2n-a-b+4)(n-a+2)}{(a-b+1)(2n-a+4)(2n-a-b+3)(n-a+1)}$

group of automorphisms on H preserving J_H and σ_H is the Lie group Sp(1). In other words, (H, J_H, σ_H) is the natural Sp(1)-module. We set $\{h_A\}_{A=\pm1}$ as a symplectic unitary basis of H and

$$\{y_{AB} := -\text{sign}(B)h_A \odot h_{-B} \mid A, B = \pm1, A+B \geq 0\}$$

as a basis of $\mathfrak{sp}(1,\mathbb{C}) \simeq S^2(H)$. We will use only the following elements and relations in the enveloping algebra $U(\mathfrak{sp}(1,\mathbb{C}))$,

$$y^0_{AB} := \delta_{A,B}, \quad y^1_{AB} := y_{AB}, \quad C_2 := \sum_{A,B} y_{AB}y_{BA}$$

and

$$y_{AB} = -\text{sign}(AB)y_{-B-A} \quad \text{for } A, B = \pm1. \tag{3.7}$$

Since all the irreducible Sp(1)-modules are parametrized by non-negative integer k, we denote by (π_k, V_k) an irreducible Sp(1)-module with highest weight k. Note that V_k is isomorphic to the kth symmetric tensor product space $S^k(H)$ of H, and $\pi_k(C_2)$ is $2k(k+2)$.

We consider $V_k \otimes H$ and decompose it, $V_k \otimes H = V_{k+1} \oplus V_{k-1}$. In the same way as (3.1), we define a linear map $p_N(\cdot)$ from V_k to V_{k+N} for $N = \pm1$ and denote by W_N the conformal weight associated to $k+N$. Here, $W_1 = -k$ and $W_{-1} = k+2$. The equation (3.2) for the Sp(1)-case is

$$\sum_{N\pm1} W_N^q p_N(h_A)^* p_N(h_B) = \text{sign}(AB)\pi_k(y^q_{-A-B}). \tag{3.8}$$

We summarize notation used in this paper.

Table 2:

	$G = \mathrm{Sp}(n)$	$G = \mathrm{Sp}(1)$
suffices	$\alpha, \beta, \nu = \pm 1, \ldots, \pm n$	$A, B, N = \pm 1$
the natural G-module	(E, J_E, σ_E)	(H, J_H, σ_H)
a (symplectic) unitary basis	$\{\epsilon_\alpha\}_\alpha$	$\{h_A\}_A$
a basis of the Lie algebra	$\{x_{\alpha\beta} \| \alpha + \beta \geq 0\}$	$\{y_{AB} \| A + B \geq 0\}$
Casimir elements	$\{c_q\}_{q \geq 0}$, $\{\hat{c}_q\}_{q \geq 0}$	C_2
irreducible G-module	(π_ρ, V_ρ)	(π_k, V_k)
the dominant integral condition	$\rho = (\rho^1, \ldots, \rho^n) \in \mathbb{Z}^n$, $\rho^1 \geq \cdots \geq \rho^n \geq 0$	$k \in \mathbb{Z}$, $k \geq 0$
linear mapping	$p_\nu(\cdot) : V_\rho \to V_{\rho+\mu_\nu}$	$p_N(\cdot) : V_k \to V_{k+N}$
conformal weights	w_ν, $\hat{w}_\nu$	W_N

4 Symbols of gradients and their relations

Let (H, J_H, σ_H) (resp. (E, J_E, σ_E)) be the natural $\mathrm{Sp}(1)$-module (resp. $\mathrm{Sp}(n)$-module). The outer tensor product space $H \hat{\otimes} E$ has a real structure $J_H \hat{\otimes} J_E$ and a Hermitian structure $\sigma_H(\cdot, J_H(\cdot)) \hat{\otimes} \sigma_E(\cdot, J_E(\cdot))$. The real part of this vector space is a real $4n$-dimensional vector space with positive inner product. This $H \hat{\otimes} E$ is a model of tangent space of a quaternionic Kähler manifold. Taking unitary bases $\{h_A\}_A$ of H and $\{\epsilon_\alpha\}_\alpha$ of E, we have the one of $H \hat{\otimes} E$,

$$\{v_{A,\alpha} := h_A \hat{\otimes} \epsilon_\alpha \mid A = \pm 1, \alpha = \pm 1, \ldots, \pm n\}.$$

We consider the Lie groups $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ and $\mathrm{Sp}(1)\mathrm{Sp}(n) := (\mathrm{Sp}(1) \times \mathrm{Sp}(n))/\{\pm I\}$. Each unitary irreducible representation of $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ is given by

$$(\pi_{k,\rho}, V_{k,\rho}) := (\pi_k \hat{\otimes} \pi_\rho, V_k \hat{\otimes} V_\rho).$$

When $k + \sum_i \rho^i$ is odd, $(\pi_{k,\rho}, V_{k,\rho})$ does not factor through a representation of $\mathrm{Sp}(1)\mathrm{Sp}(n)$. Furthermore, from quaternionic or real structures on V_k and V_ρ, we can set a quaternionic structure on $V_{k,\rho}$. When $k + \sum_i \rho^i$ is even, $V_{k,\rho}$ is an irreducible $\mathrm{Sp}(1)\mathrm{Sp}(n)$-module with real structure. For example, $H \hat{\otimes} E = V_1 \hat{\otimes} V_{(1_1)}$ is an irreducible $\mathrm{Sp}(1)\mathrm{Sp}(n)$-module with real structure $J_H \hat{\otimes} J_E$.

Now, we consider a representation space $V_{k,\rho} \otimes (H \hat{\otimes} E)$ of $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ or $\mathrm{Sp}(1)\mathrm{Sp}(n)$ and decompose it,

$$V_{k,\rho} \otimes (H \hat{\otimes} E) = \bigoplus_{\substack{N=\pm 1, \\ \nu=\pm 1,\ldots,\pm n}} V_{k+N,\rho+\mu_\nu}.$$

Considering the orthogonal projection from $V_{k,\rho} \otimes (H\hat{\otimes}E)$ onto $V_{k+N,\rho+\mu_\nu}$, we define a linear mapping $p_{N,\nu}(\cdot)$ from $V_{k,\rho}$ to $V_{k+N,\rho+\mu_\nu}$ by

$$p_{N,\nu}(h\hat{\otimes}\epsilon)(\phi\hat{\otimes}\psi) := p_N(h)\phi\hat{\otimes}p_\nu(\epsilon)\psi$$

for $h\hat{\otimes}\epsilon$ in $H\hat{\otimes}E$ and $\phi\hat{\otimes}\psi$ in $V_{k,\rho} = V_k\hat{\otimes}V_\rho$. The adjoint map of $p_{N,\nu}(h\hat{\otimes}\epsilon)$ is defined to be $p_N(h)^*\hat{\otimes}p_\nu(\epsilon)^*$. These maps are just the principal symbols of first order differential operators called quaternionic Kählerian gradients given in the next section. As mentioned in Section 1, to obtain Bochner-Weitzenböck formulas for the operators, we need anti-symmetric relations among the principal symbols. Such relations follow from the next proposition.

Proposition 4.1. *The linear maps $\{p_{N,\nu}(v_{A,\alpha})\}_{N,\nu,A,\alpha}$ on $V_{k,\rho}$ satisfy the following.*

1.

$$\begin{aligned}&\sum_{N,\nu} \hat{w}_\nu^q p_{N,\nu}(v_{A,\alpha})^* p_{N,\nu}(v_{B,\beta}) \\ =&\mathrm{sign}(AB\alpha\beta) \sum_{N,\nu} \left\{ (-1)^q \hat{w}_\nu^q - \frac{1-(-1)^q}{2} \hat{w}_\nu^{q-1} \right. \\ &\left. - \sum_{p=0}^{q-1} (-1)^p \pi_\rho(\hat{c}_{q-1-p}) \hat{w}_\nu^p \right\} p_{N,\nu}(v_{-B,-\beta})^* p_{N,\nu}(v_{-A,-\alpha}).\end{aligned} \tag{4.1}$$

2. When k is not zero,

$$\begin{aligned}&\sum_{N,\nu} W_N \hat{w}_\nu^q p_{N,\nu}(v_{A,\alpha})^* p_{N,\nu}(v_{B,\beta}) \\ =&-\mathrm{sign}(AB\alpha\beta) \sum_{N,\nu} W_N \left\{ (-1)^q \hat{w}_\nu^q - \frac{1-(-1)^q}{2} \hat{w}_\nu^{q-1} \right. \\ &\left. - \sum_{p=0}^{q-1} (-1)^p \pi_\rho(\hat{c}_{q-1-p}) \hat{w}_\nu^p \right\} p_{N,\nu}(v_{-B,-\beta})^* p_{N,\nu}(v_{-A,-\alpha}).\end{aligned} \tag{4.2}$$

Proof. It follows from (3.3) and (3.8) that

$$\sum_{N,\nu} \hat{w}_\nu^q p_{N,\nu}(v_{A,\alpha})^* p_{N,\nu}(v_{B,\beta}) = \mathrm{sign}(AB\alpha\beta)\delta_{A,B}\mathrm{id}\hat{\otimes}\pi_\rho(\hat{x}^q_{-\alpha-\beta}), \tag{4.3}$$

$$\sum_{N,\nu} W_N \hat{w}_\nu^q p_{N,\nu}(v_{A,\alpha})^* p_{N,\nu}(v_{B,\beta}) = \mathrm{sign}(AB\alpha\beta)\pi_k(y_{-A-B})\hat{\otimes}\pi_\rho(\hat{x}^q_{-\alpha-\beta}).$$

Substituting (2.1) and (3.7) for the above equations, we can prove the proposition. □

5 Gradients on quaternionic Kähler manifolds

Let (M, g) be a real $4n$-dimensional quaternionic Kähler manifold. The frame bundle of M reduces to a principal bundle $\mathbf{P}$ with structure group $\mathrm{Sp}(1)\mathrm{Sp}(n)$. Take the $\mathrm{Sp}(1)\mathrm{Sp}(n)$-module $H\hat{\otimes}E$, and we have an associated vector bundle $\mathbf{H}\hat{\otimes}\mathbf{E} := \mathbf{P} \times_{\mathrm{Sp}(1)\mathrm{Sp}(n)} (H\hat{\otimes}E)$ with real structure and Hermitian metric. The real part of $\mathbf{H}\hat{\otimes}\mathbf{E}$ is isometric to the tangent bundle $T(M)$.

For an irreducible $\mathrm{Sp}(1)\mathrm{Sp}(n)$-module $V_{k,\rho}$, we have an associated vector bundle $\mathbf{S}_{k,\rho} := \mathbf{P} \times_{\mathrm{Sp}(1)\mathrm{Sp}(n)} V_{k,\rho}$. Since the Levi-Civita connection reduces a connection on $\mathbf{P}$, we have a covariant derivative ∇ on $\mathbf{S}_{k,\rho}$,

$$\nabla : \Gamma(M, \mathbf{S}_{k,\rho}) \to \Gamma(M, \mathbf{S}_{k,\rho} \otimes (T^*(M) \otimes \mathbb{C})) \simeq \Gamma(M, \mathbf{S}_{k,\rho} \otimes (\mathbf{H}\hat{\otimes}\mathbf{E})).$$

Here we identified $T^*(M) \otimes \mathbb{C}$ with $T(M) \otimes \mathbb{C} \simeq \mathbf{H}\hat{\otimes}\mathbf{E}$ by complex inner product $\sigma_H \hat{\otimes} \sigma_E$. Decomposing ∇ with respect to $\mathrm{Sp}(1)\mathrm{Sp}(n)$, we have first order differential operators in the following way. Let $\{h_A\}_A$ and $\{\epsilon_\alpha\}_\alpha$ be local unitary frames of $\mathbf{H}$ and $\mathbf{E}$, respectively. For a smooth section ϕ of $\mathbf{S}_{k,\rho}$, the derivative $\nabla\phi$ is locally expressed by

$$\nabla\phi = \sum_{A,\alpha} \nabla_{v_{A,\alpha}} \phi \otimes v^*_{A,\alpha} = \sum_{A,\alpha} \mathrm{sign}(A\alpha) \nabla_{v_{A,\alpha}} \phi \otimes v_{-A,-\alpha},$$

where we set $v_{A,\alpha} := h_A \hat{\otimes} \epsilon_\alpha$. We project $\nabla\phi$ from $\mathbf{S}_{k,\rho} \otimes (\mathbf{H}\hat{\otimes}\mathbf{E})$ onto an irreducible bundle $\mathbf{S}_{k+N,\rho+\mu_\nu}$ fiberwise. Then we define a first order differential operator $D_{N,\nu} : \Gamma(M, \mathbf{S}_{k,\rho}) \to \Gamma(M, \mathbf{S}_{k+N,\rho+\mu_\nu})$ by

$$D_{N,\nu} := \sum_{A,\alpha} \mathrm{sign}(A\alpha) p_{N,\nu}(v_{-A,-\alpha}) \nabla_{v_{A,\alpha}}. \tag{5.1}$$

It is easy to show that the formal adjoint operator $(D_{N\nu})^*$ of $D_{N\nu}$ is given by

$$(D_{N,\nu})^* = -\sum p_{N,\nu}(v_{A,\alpha})^* \nabla_{v_{A,\alpha}} : \Gamma(M, \mathbf{S}_{k+N,\rho+\mu_\nu}) \to \Gamma(M, \mathbf{S}_{k,\rho}).$$

If $k + N$ or $\rho + \mu_\nu$ does not satisfy dominant integral condition, we set $\mathbf{S}_{k+N,\rho+\mu_\nu} := M \times \{0\}$ and $D_{N,\nu} := 0$ virtually. We call these operators $\{D_{N,\nu}, (D_{N,\nu})^*\}_{N,\nu}$ *gradients on a quaternionic Kähler manifold* or *quaternionic Kählerian gradients*. Most of first order differential operators in quaternionic Kähler geometry are realized as gradients.

Remark 5.1. The obstruction for lifting $\mathbf{P}$ to a principal $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ bundle $\tilde{\mathbf{P}}$ is the second Stiefel-Whitney class of the real part of $S^2(\mathbf{H})$ (cf. [15]). When the obstruction is zero, we can consider a vector bundle $\mathbf{S}_{k,\rho}$ associated to $\tilde{\mathbf{P}}$. In this case, we have gradients even if $k + \sum \rho^i$ is odd. Since our results come from local calculation, we can do well in the case that $k + \sum \rho^i$ is odd.

Example 5.1. Let M be a spin manifold and $\mathbf{S}(M)$ be the spinor bundle. The Dirac operator D is realized as an irreducible component of ∇ on $\mathbf{S}(M)$,

$$D : \Gamma(M, \mathbf{S}(M)) \xrightarrow{\nabla} \Gamma(M, \mathbf{S}(M) \otimes (T^*(M) \otimes \mathbb{C})) \xrightarrow{\Pi} \Gamma(M, \mathbf{S}(M)).$$

If M has a quaternionic Kähler structure, then the spinor bundle $\mathbf{S}(M)$ is decomposed with respect to $\mathrm{Sp}(1)\mathrm{Sp}(n)$ or $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$,

$$\mathbf{S}(M) = \bigoplus_{k=0}^{n} \mathbf{S}_{k,(1_{n-k})} = \bigoplus_{k=0}^{n} S^k(\mathbf{H}) \hat{\otimes} \Lambda_0^{n-k}(\mathbf{E}).$$

We divide the Dirac operator D along this decomposition. Then each piece of D is a quaternionic Kählerian gradient (cf. [10]).

In the rest of this section, we show a conformal covariance of gradient. An almost quaternionic Hermitian manifold (M, g) is a $4n$-dimensional Riemannian manifold whose frame bundle reduces to an $\mathrm{Sp}(1)\mathrm{Sp}(n)$-bundle $\mathbf{P}$. Though the Levi-Civita connection is not always a connection on $\mathbf{P}$, we can project it onto $\mathbf{P}$ and obtain a connection ω on $\mathbf{P}$. In other words, ω is the $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$-part of the Levi-Civita connection (cf. [6]). Note that M is a quaternionic Kähler manifold if the torsion tensor of ω is zero. In the same manner as (5.1), we construct first order differential operators $\{D_{N,\mu}\}_{N,\nu}$ on $\mathbf{S}_{k,\rho}$ with respect to ω. We also call them gradients.

Let (M, g) be a quaternionic Kähler manifold. A conformal deformation $g' := e^{2\sigma(x)} g$ of the Riemannian metric g gives an almost quaternionic Hermitian manifold (M, g'). Then we have two gradients, $D_{N,\nu}$ on (M, g) and $D'_{N,\nu}$ on (M, g'). The next proposition is the reason why w_ν and W_N are called conformal weights.

Proposition 5.1. *The gradient $D_{N,\nu}$ on (M, g) is related to $D'_{N,\nu}$ on $(M, g') = (M, e^{2\sigma} g)$ covariantly,*

$$D'_{N,\nu} = \exp\left(\left(-\frac{w_\nu}{2} - \frac{W_N}{2n} - 1\right)\sigma(x)\right) \circ D_{N,\nu} \circ \exp\left(\left(\frac{w_\nu}{2} + \frac{W_N}{2n}\right)\sigma(x)\right).$$

Proof. In [6], the author showed a conformal covariance of gradients on a hyper-Kähler manifold. In the same way, we can prove the above conformal covariance of quaternionic Kählerian gradients. □

6 Bochner-Weitzenböck formulas

We consider a vector bundle $\mathbf{S}_{k,\rho}$ on a quaternionic Kähler manifold M. Set $B_{N,\nu} := (D_{N,\nu})^* D_{N,\nu}$, and we know that the second order operator has the following expression,

$$B_{N,\nu} = (D_{N,\nu})^* D_{N,\nu} = -\sum_{A,B,\alpha,\beta} \mathrm{sign}(A\alpha) p_{N,\nu}(v_{B,\beta})^* p_{N,\nu}(v_{A,\alpha}) \nabla^2_{v_{B,\beta}, v_{-A,-\alpha}},$$

where $\nabla^2_{X,Y}$ is defined to be $\nabla_X \nabla_Y - \nabla_{\nabla_X Y}$ for vector fields X and Y. It follows from (4.3) of $q = 0$ that

$$\sum_{N,\nu} B_{N,\nu} = \nabla^* \nabla = -\sum_{A,\alpha} \mathrm{sign}(A\alpha) \nabla^2_{v_{A,\alpha}, v_{-A,-\alpha}},$$

where $\nabla^*\nabla$ is the connection Laplacian on $\mathbf{S}_{k,\rho}$. Thus a linear combination of $\{B_{N,\nu}\}_{N,\nu}$ has second order in general. But, some appropriate combinations are curvature endomorphisms which are zeroth order operators, that is,

$$\sum_{N,\nu} a_{N,\nu} B_{N,\nu} = (\text{curvature endomorphism}).$$

We call such equations *(optimal) Bochner-Weitzenböck formulas* (see [2]).

Let us give Bochner-Weitzenböck formulas for quaternionic Kählerian gradients. We define the curvature $R_{k,\rho}$ of ∇ on $\mathbf{S}_{k,\rho}$ by

$$R_{k,\rho}(X,Y) := \nabla^2_{X,Y} - \nabla^2_{Y,X}.$$

Then, from (4.1) and (4.2), we have

$$\begin{aligned} &\sum_{N,\nu} \left\{ (1-(-1)^q)\hat{w}^q_\nu + \frac{1-(-1)^q}{2}\hat{w}^{q-1}_\nu + \sum_{p=0}^{q-1} (-1)^p \pi_\rho(\hat{c}_{q-1-p})\hat{w}^p_\nu \right\} B_{N,\nu} \\ &= -\sum_{A,\alpha,\beta} \mathrm{sign}(A\alpha) \mathrm{id} \hat{\otimes} \pi_\rho(\hat{x}^q_{-\alpha\beta}) R_{k,\rho}(v_{A,\alpha}, v_{-A,\beta}), \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} &\sum_{N,\nu} W_N \left\{ (1+(-1)^q)\hat{w}^q_\nu - \frac{1-(-1)^q}{2}\hat{w}^{q-1}_\nu - \sum_{p=0}^{q-1} (-1)^p \pi_\rho(\hat{c}_{q-1-p})\hat{w}^p_\nu \right\} B_{N,\nu} \\ &= -\sum_{A,B,\alpha,\beta} \mathrm{sign}(A\alpha) \pi_k(y_{-AB}) \hat{\otimes} \pi_\rho(\hat{x}^q_{-\alpha\beta}) R_{k,\rho}(v_{A,\alpha}, v_{B,\beta}). \end{aligned} \tag{6.2}$$

To obtain more simple formulas, we should calculate curvature endomorphisms in the above equations. We denote by R the Riemannian curvature tensor on the tangent bundle $T(M)$. In [10], W. Kramer, U. Semmelmann and G. Weingart gave a formula of R as follows. We define three $\mathrm{End}(\mathbf{H}\hat{\otimes}\mathbf{E})$-valued 2-forms by

$$\begin{aligned}R^H(h\hat{\otimes}\epsilon, h'\hat{\otimes}\epsilon') &:= \sigma_E(\epsilon,\epsilon')(h\odot h'\otimes \mathrm{id}_E),\\ R^E(h\hat{\otimes}\epsilon, h'\hat{\otimes}\epsilon') &:= \sigma_H(h,h')(\mathrm{id}_H\otimes \epsilon\odot\epsilon'),\\ R^{hyper}(h\hat{\otimes}\epsilon, h'\hat{\otimes}\epsilon') &:= \sigma_H(h,h')(\mathrm{id}_E\hat{\otimes}\mathfrak{R}(\epsilon,\epsilon')).\end{aligned}$$

Here, $\mathfrak{R}$ is the $S^4(\mathbf{E})$-part of the curvature R. In other words, $\sigma_E(\mathfrak{R}(\epsilon^1,\epsilon^2)\epsilon^3,\epsilon^4)$ is symmetric for $\epsilon^1,\epsilon^2,\epsilon^3$ and ϵ^4. Note that $\mathfrak{R}(\epsilon,\epsilon')$ is expressed by

$$\mathfrak{R}(\epsilon,\epsilon') = 1/2\sum_{\delta,\gamma}\mathrm{sign}(\delta)\sigma_E(\mathfrak{R}(\epsilon,\epsilon')\epsilon_\gamma,\epsilon_{-\delta})x_{\delta\gamma}.$$

Then the Riemannian curvature tensor R is

$$R = -\frac{\kappa}{8n(n+2)}\left(R^H + R^E\right) + R^{hyper}.$$

Here, κ is the scalar curvature of R. Since a quaternionic Kähler manifold is an Einstein manifold, the scalar curvature κ is constant. Note that R^{hyper} is zero on the quaternionic projective space $\mathbb{H}P^n$ (see [15]).

The covariant derivative ∇ on $\mathbf{S}_{k,\rho}$ is defined from the Levi-Civita connection and hence the curvature $R_{k,\rho}$ is $\pi_{k,\rho}(R)$. The curvature endomorphisms on the right sides of (6.1) and (6.2) are rewritten as follows.

Lemma 6.1. *We set*

$$\hat{\mathfrak{R}}^q_\rho := \sum_{\alpha,\beta}\mathrm{sign}(\alpha)\mathrm{id}\hat{\otimes}\pi_\rho(\hat{x}^q_{\alpha\beta}\mathfrak{R}(\epsilon_{-\alpha},\epsilon_\beta)).$$

Then we have

$$\begin{aligned}&-\sum_{A,\alpha,\beta}\mathrm{sign}(A\alpha)\mathrm{id}\hat{\otimes}\pi_\rho(\hat{x}^q_{-\alpha\beta})R_{k,\rho}(v_{A,\alpha},v_{-A,\beta})\\ =&\frac{\kappa}{4n(n+2)}\pi_\rho\left(\hat{c}_{q+1}+\frac{2n+1}{2}\hat{c}_q\right)+2\hat{\mathfrak{R}}^q_\rho,\end{aligned}$$

and

$$\begin{aligned}&-\sum_{A,B,\alpha,\beta}\mathrm{sign}(A\alpha)\pi_k(y_{-AB})\hat{\otimes}\pi_\rho(\hat{x}^q_{-\alpha\beta})R_{k,\rho}(v_{A,\alpha},v_{B,\beta})\\ =&\frac{\kappa}{8n(n+2)}\pi_k(C_2)\pi_\rho(\hat{c}_q) = \frac{k(k+2)\kappa}{4n(n+2)}\pi_\rho(\hat{c}_q).\end{aligned}$$

We detect the number of independent Bochner-Weitzenböck formulas. We assume that there are $\mathcal{N}$ irreducible components in $\mathbf{S}_{k,\rho} \otimes (\mathbf{H}\hat{\otimes}\mathbf{E})$, that is,

$$\mathcal{N} := \#\{(k+N, \rho+\mu_\nu) \mid \text{both } k+N \text{ and } \rho+\mu_\nu \text{ are dominant integral}\}.$$

Then we have $\mathcal{N}$ gradients on $\Gamma(M, \mathbf{S}_{k,\rho})$. From a similar discussion to the one in [8], we can show that there are at least $[\mathcal{N}/2]$ independent Bochner-Weitzenböck formulas.

Theorem 6.2. *We assume that the number of non-zero gradients is $\mathcal{N}$. The operators $\{B_{N,\nu} = (D_{N,\nu})^* D_{N\nu}\}_{N,\nu}$ on $\Gamma(M, \mathbf{S}_{k,\rho})$ satisfy*

$$\sum_{N,\nu} B_{N,\nu} = \nabla^*\nabla.$$

Furthermore, when the highest weight k for Sp(1) *is not zero, we have the following $[\mathcal{N}/2]$ independent Bochner-Weitzenböck formulas:*

1. *For $q = 1, 2, \dots, [\mathcal{N}/4]$,*

$$\begin{aligned} &\sum_{N,\nu} \left\{ \sum_{p=0}^{2q-1} (-1)^p \pi_\rho(\hat{c}_{2q-1-p}) \hat{w}_\nu^p \right\} B_{N,\nu} \\ =&\frac{\kappa}{4n(n+2)} \pi_\rho \left(\hat{c}_{2q+1} + \frac{2n+1}{2} \hat{c}_{2q} \right) + 2\hat{\mathfrak{R}}_\rho^{2q}. \end{aligned} \tag{6.3}$$

2. *For $q = 0, 1, \dots, [\mathcal{N}/4 - 1/2]$.*

$$\sum_{N,\nu} W_N \left\{ 2\hat{w}_\nu^{2q} - \sum_{p=0}^{2q-1} (-1)^p \pi_\rho(\hat{c}_{2q-1-p}) \hat{w}_\nu^p \right\} B_{N,\nu} = \frac{k(k+2)\kappa}{4n(n+2)} \pi_\rho(\hat{c}_{2q}). \tag{6.4}$$

When k is zero, the equations (6.3) *for $q = 1, \dots, [\mathcal{N}/2]$ constitute $[\mathcal{N}/2]$ independent Bochner-Weitzenböck formulas.*

It is useful to write down the first few formulas. From (2.1)–(2.3), (3.5) and (3.6), we rewrite (6.3) of $q = 1, 2$ as

$$\sum_{N,\nu} w_\nu B_{N,\nu} = \frac{\kappa}{8n(n+2)} \pi_\rho(c_2) + \sum_{\alpha,\beta} \operatorname{sign}(\alpha) \mathrm{id} \hat{\otimes} \pi_\rho(x_{\alpha\beta} \mathfrak{R}(\epsilon_{-\alpha}, \epsilon_\beta)), \tag{6.5}$$

$$\begin{aligned} &\sum_{N,\nu} \left\{ \pi_\rho(c_2)/2 + (n+1)(2n+1) w_\nu - (2n+1) w_\nu^2 + w_\nu^3 \right\} B_{N,\nu} \\ &= \frac{\kappa}{8n(n+2)} \pi_\rho(c_4) + \sum_{\alpha,\beta} \operatorname{sign}(\alpha) \mathrm{id} \hat{\otimes} \pi_\rho(x_{\alpha\beta}^3 \mathfrak{R}(\epsilon_{-\alpha}, \epsilon_\beta)), \end{aligned} \tag{6.6}$$

and (6.4) of $q = 0, 1, 2$ as

$$\sum_{N,\nu} W_N B_{N,\nu} = \frac{k(k+2)\kappa}{4(n+2)}, \tag{6.7}$$

$$\sum_{N,\nu} 2W_N \left(w_\nu^2 - (n+1)w_\nu\right) B_{N,\nu} = \frac{k(k+2)\kappa}{4n(n+2)}\pi_\rho(c_2), \tag{6.8}$$

$$\begin{aligned}\sum_{N,\nu} W_N \{&2w_\nu(w_\nu - n - 1)(w_\nu^2 - (2n+1)w_\nu + 2n + 1) \\ &+ (n + w_\nu)\pi_\rho(c_2)\} B_{N,\nu} = \frac{k(k+2)\kappa}{4n(n+2)}\pi_\rho(c_4).\end{aligned} \tag{6.9}$$

To apply Bochner-Weitzenböck formulas to differential geometry, we compare the curvature endomorphisms for the Riemannian case and the ones for the quaternionic Kähler case. In [5] and [8], the author discussed the Bochner-Weitzenböck formulas and associated curvature endomorphisms on a Riemannian or spin manifold. Let (M, g) be an oriented m-dimensional Riemannian manifold. We take an irreducible vector bundle $\mathbf{S}_\lambda$ associated to the orthonormal frame bundle of M, where λ is a dominant integral weight with respect to $\mathrm{SO}(m)$. Then we define a curvature endomorphism R^1_λ on $\mathbf{S}_\lambda$ by

$$R^1_\lambda = \sum_{1 \le i,j \le m} \pi_\lambda(e_i \wedge e_j) R_\lambda(e_i, e_j),$$

where $\{e_i\}_{1 \le i \le m}$ is a local orthonormal frame of M and R_λ is the curvature on $\mathbf{S}_\lambda$. When M has a spin structure, we can also define a curvature endomorphism R^1_λ on vector bundle $\mathbf{S}_\lambda$ associated to the bundle of spin frames.

If M has a quaternionic Kähler structure, we can decompose $\mathbf{S}_\lambda$ into irreducible bundles with respect to $\mathrm{Sp}(1)\mathrm{Sp}(n) \subset \mathrm{SO}(4n)$, $\mathbf{S}_\lambda = \bigoplus_{k,\rho} \mathbf{S}_{k,\rho}$. Then it is easily to show that the restriction of R^1_λ onto $\mathbf{S}_{k,\rho}$ is given by

$$R^1_{k,\rho} := \frac{\kappa}{8n(n+2)}\left(\pi_k(C_2) + \pi_\rho(c_2)\right) + \sum_{\alpha,\beta} \mathrm{sign}(\alpha)\mathrm{id}\hat{\otimes}\pi_\rho(x_{\alpha\beta}\mathfrak{R}(\epsilon_{-\alpha}, \epsilon_\beta)).$$

From (6.5) and (6.7), we have a quaternionic Kählerian version of Gauduchon's formula in [4],

$$R^1_{k,\rho} = R^1_\lambda|_{\mathbf{S}_{k,\rho}} = \sum_{N,\nu} (w_\nu + W_N/n) B_{N,\nu}.$$

Remark 6.1. The curvature endomorphism $R^1_{k,\rho}$ corresponds to $4q(R)$ in [16].

Example 6.1. We consider the bundle $\mathbf{S}_\lambda = \Lambda^p(M)$. Then we have a Bochner-Weitzenböck formula for the Laplace operator (see [8]),

$$dd^* + d^*d = \nabla^*\nabla + R^1_\lambda/2.$$

Restricting the above equation onto an irreducible bundle $\mathbf{S}_{k,\rho}$, we have

$$dd^* + d^*d = \nabla^*\nabla + R^1_{k,\rho}/2 = \sum_{N,\nu}\left(1 + \frac{w_\nu}{2} + \frac{W_N}{2n}\right) B_{N,\nu}.$$

Example 6.2. We consider the quaternionic projective space $\mathbb{H}P^n$ with $\kappa = 2n$. Since the curvature $R^{hyper} = 0$, we obtain

$$dd^* + d^*d = \nabla^*\nabla + R^1_{k,\rho}/2 = \nabla^*\nabla + \frac{1}{8(n+2)}(2k(k+2) + \pi_\rho(c_2)).$$

Example 6.3. Let M be a quaternionic Kähler spin manifold and $\mathbf{S}_\lambda$ be the spinor bundle. The Dirac operator satisfies

$$D^2 = \nabla^*\nabla + R^1_\lambda = \nabla^*\nabla + \kappa/4.$$

With quaternionic Kähler structure of M, we decompose the spinor bundle as

$$\mathbf{S}_\lambda = \bigoplus_{1\le k\le n} \mathbf{S}_{k,(1_{n-k})} = \bigoplus_{1\le k\le n} S^k(\mathbf{H})\hat{\otimes}\Lambda^{n-k}_0(\mathbf{E}).$$

The restriction of D^2 to $\mathbf{S}_{k,(1_{n-k})}$ is

$$D^2 = \nabla^*\nabla + \kappa/4 = \nabla^*\nabla + R^1_{k,(1_{n-k})} = \sum_{N,\nu}(1 + w_\nu + W_N/n)B_{N,\nu}.$$

7 Clifford algebras and curvature endomorphisms on differential forms

We define the curvature endomorphisms $\{\mathfrak{R}^q_\rho\}_{q\ge 0}$ by

$$\begin{aligned}\mathfrak{R}^q_\rho :=& \sum_{\alpha,\beta} \mathrm{sign}(\alpha)\mathrm{id}\hat{\otimes}\pi_\rho(x^q_{\alpha\beta}\mathfrak{R}(\epsilon_{-\alpha}, \epsilon_\beta)) \\ =& 1/2 \sum_{\alpha,\beta,\delta,\gamma} \mathrm{sign}(\alpha\delta)\sigma_E(\mathfrak{R}(\epsilon_{-\alpha}, \epsilon_\beta)\epsilon_\gamma, \epsilon_{-\delta})\mathrm{id}\hat{\otimes}\pi_\rho(x^q_{\alpha\beta}x_{\delta\gamma}).\end{aligned}$$

We shall investigate the above endomorphisms on $\Lambda^p(M)$ more precisely. It is known that each irreducible summand in $\Lambda^p(M)$ is the form of $\mathbf{S}_{k,(2_b,1_{a-b})} =$

$S^k(\mathbf{H})\hat{\otimes}\Lambda_0^{a,b}(\mathbf{E})$ for $0 \leq k \leq 2n-a-b$ and $0 \leq b \leq a \leq n$ (cf. [16]). Since the curvature endomorphism $\mathfrak{R}^q_\rho$ is independent of the Sp(1)-part, we may consider only $\mathbf{S}_{0,(2_b,1_{a-b})} = \Lambda_0^{a,b}(\mathbf{E})$. The main result in this section is the following.

Proposition 7.1. *On $\Lambda_0^{a,b}(\mathbf{E})$, we have*

$$\mathfrak{R}^3_{(2_b,1_{a-b})} = (2n^2+7n+7-\pi_{(2_b,1_{a-b})}(c_2)/4)\mathfrak{R}^1_{(2_b,1_{a-b})}. \tag{7.1}$$

Furthermore, we have $\mathfrak{R}^1_{(1_a)} = 0$ on $\Lambda_0^a(\mathbf{E}) = \Lambda_0^{a,0}(\mathbf{E})$.

Our method is to make use of the Clifford algebra $\mathrm{Cl}(E)$ associated to (E, σ_E, J_E). Let $\{\epsilon_\alpha\}_\alpha$ be a symplectic unitary basis of E. The Clifford algebra $\mathrm{Cl}(E)$ is an associated algebra over $\mathbb{C}$ generated by $\{\epsilon_\alpha, \epsilon_\alpha^\dagger\}_\alpha \cup \{1\}$ with relations

$$\epsilon_\alpha\epsilon_\beta + \epsilon_\beta\epsilon_\alpha = 0, \quad \epsilon_\alpha^\dagger\epsilon_\beta^\dagger + \epsilon_\beta^\dagger\epsilon_\alpha^\dagger = 0, \quad \epsilon_\alpha\epsilon_\beta^\dagger + \epsilon_\beta^\dagger\epsilon_\alpha = \delta_{\alpha,\beta}.$$

This algebra acts on $\bigoplus_{a=0}^n \Lambda^a(E)$ by

$$\epsilon_\alpha^\dagger \cdot := i(\epsilon_\alpha), \quad \epsilon_\alpha \cdot := \epsilon_\alpha \wedge,$$

where $i(\epsilon_\alpha)$ is the interior product and $\epsilon_\alpha\wedge$ is the exterior product. Then the representation $(\pi_a, \Lambda^a(E))$ of $\mathfrak{sp}(n,\mathbb{C})$ is realized by

$$\pi_a(x_{\alpha\beta}) = \epsilon_\alpha\epsilon_\beta^\dagger - \mathrm{sign}(\alpha\beta)\epsilon_{-\beta}\epsilon_{-\alpha}^\dagger$$

for $x_{\alpha\beta}$ in $\mathfrak{sp}(n,\mathbb{C})$. In other words, the Lie algebra $\mathfrak{sp}(n,\mathbb{C})$ is embedded in $\mathrm{Cl}(E)$ by $x_{\alpha\beta} \mapsto \epsilon_\alpha\epsilon_\beta^\dagger - \mathrm{sign}(\alpha\beta)\epsilon_{-\beta}\epsilon_{-\alpha}^\dagger$.

Remark 7.1. There are $\mathfrak{sp}(n,\mathbb{C})$-invariant elements in $\mathrm{Cl}(E)$,

$$\mathbf{N} := \sum_\alpha \epsilon_\alpha\epsilon_\alpha^\dagger, \quad \mathbf{N}^\dagger := \sum_\beta \epsilon_\alpha^\dagger\epsilon_\alpha,$$
$$\sigma_E := \sum \mathrm{sign}(\alpha)\epsilon_\alpha\epsilon_{-\alpha}, \quad \sigma_E^\dagger := \sum \mathrm{sign}(\alpha)\epsilon_\alpha^\dagger\epsilon_{-\alpha}^\dagger.$$

The operator $\mathbf{N}$ is the so-called number operator acting on $\Lambda^a(E)$ by constant a, and $\mathbf{N}^\dagger = 2n - \mathbf{N}$. The operator σ_E is used for decomposing $\Lambda^a(E)$. More precisely, we have $[\sigma_E, \sigma_E^\dagger] = 4n - 4a$ on $\Lambda^a(E)$, and $\Lambda^a(E) = \bigoplus_{p=0}^{[a/2]} \sigma_E^p \Lambda_0^{a-2p}(E)$. We use these invariant operators implicitly in the proof below.

Proof of Proposition 7.1. Though the second claim that $\mathfrak{R}^1_{(1_a)}$ is zero has been already shown in [10], we give a proof of it as a good exercise before proving the first claim.

It follows from the Clifford relations and the symmetry of $\sigma_E(\mathfrak{R}(\cdot,\cdot)\cdot,\cdot)$ that

$$\begin{aligned}
&\sum_{\alpha} \mathrm{sign}(\alpha)\pi_a(x_{\alpha\beta}\mathfrak{R}(\epsilon_{-\alpha},\epsilon_\beta))\\
=&1/2\sum_{\alpha,\beta,\delta,\gamma} \mathrm{sign}(\alpha\delta)\sigma_E(\mathfrak{R}(\epsilon_{-\alpha},\epsilon_\beta)\epsilon_\gamma,\epsilon_{-\delta})\pi_a(x_{\alpha\beta}x_{\delta\gamma})\\
=&1/2\sum_{\alpha,\beta,\delta,\gamma} \mathrm{sign}(\alpha\delta)\sigma_E(\mathfrak{R}(\epsilon_{-\alpha},\epsilon_\beta)\epsilon_\gamma,\epsilon_{-\delta})\\
&\qquad(\epsilon_\alpha\epsilon_\beta^\dagger-\mathrm{sign}(\alpha\beta)\epsilon_{-\beta}\epsilon_{-\alpha}^\dagger)(\epsilon_\delta\epsilon_\gamma^\dagger-\mathrm{sign}(\delta\gamma)\epsilon_{-\gamma}\epsilon_{-\delta}^\dagger)\\
=&1/2\sum_{\alpha,\beta,\delta,\gamma} \mathrm{sign}(\alpha\delta)\sigma_E(\mathfrak{R}(\epsilon_{-\alpha},\epsilon_\beta)\epsilon_\gamma,\epsilon_{-\delta})\\
&\{\mathrm{sign}(\beta\delta)(\delta_{\beta,\delta}\epsilon_\alpha\epsilon_\gamma^\dagger+\delta_{-\alpha,\delta}\epsilon_{-\beta}\epsilon_\gamma^\dagger+\delta_{\beta,-\gamma}\epsilon_\alpha\epsilon_{-\delta}^\dagger+\delta_{\alpha,\gamma}\epsilon_{-\beta}\epsilon_{-\delta}^\dagger)\\
&\quad-\epsilon_\alpha\epsilon_\delta\epsilon_\beta^\dagger\epsilon_\gamma^\dagger+\mathrm{sign}(\alpha\beta)\epsilon_{-\beta}\epsilon_\delta\epsilon_{-\alpha}^\dagger\epsilon_\gamma^\dagger\\
&\qquad+\mathrm{sign}(\delta\gamma)\epsilon_\alpha\epsilon_{-\gamma}\epsilon_\beta^\dagger\epsilon_{-\delta}^\dagger-\mathrm{sign}(\alpha\beta\delta\gamma)\epsilon_{-\beta}\epsilon_{-\gamma}\epsilon_{-\alpha}^\dagger\epsilon_{-\delta}^\dagger\}\\
=&0.
\end{aligned}$$

Thus we have proved the second claim, $\mathfrak{R}^1_{(1_a)}=0$. To prove the first claim, we consider the tensor representation $(\pi_a\otimes\pi_b,\Lambda^a(E)\otimes\Lambda^b(E))$. From a tedious calculation, we show

$$\begin{aligned}
&(\pi_a\otimes\pi_b)\left(\sum_{\alpha,\beta}\mathrm{sign}(\alpha)x^3_{\alpha\beta}\mathfrak{R}(\epsilon_{-\alpha},\epsilon_\beta)\right)\\
=&2(2n^2+7n+7-(\pi_a\otimes\pi_b)(c_2/4))\left(\sum_{\alpha,\beta}\mathrm{sign}(\alpha)\pi_a(\mathfrak{R}(\epsilon_{-\alpha},\epsilon_\beta))\otimes\pi_b(x_{\alpha\beta})\right).
\end{aligned}$$

On the other hand, we have

$$(\pi_a\otimes\pi_b)\left(\sum_{\alpha,\beta}\mathrm{sign}(\alpha)x_{\alpha\beta}\mathfrak{R}(\epsilon_{-\alpha},\epsilon_\beta)\right)=2\sum_{\alpha,\beta}\mathrm{sign}(\alpha)\pi_a(\mathfrak{R}(\epsilon_{-\alpha},\epsilon_\beta))\otimes\pi_b(x_{\alpha\beta}).$$

Then we conclude that

$$
\begin{aligned}
&(\pi_a \otimes \pi_b)\left(\sum_{\alpha,\beta} \operatorname{sign}(\alpha) x^3_{\alpha\beta} \mathfrak{R}(\epsilon_{-\alpha}, \epsilon_\beta)\right) \\
=& (2n^2 + 7n + 7 - (\pi_a \otimes \pi_b)(c_2/4))(\pi_a \otimes \pi_b)\left(\sum_{\alpha,\beta} \operatorname{sign}(\alpha) x_{\alpha\beta} \mathfrak{R}(\epsilon_{-\alpha}, \epsilon_\beta)\right).
\end{aligned}
$$

Restricting this equation onto $\Lambda^{a,b}_0(\mathbf{E})$ in $\Lambda^a(\mathbf{E}) \otimes \Lambda^b(\mathbf{E})$, we have proved the first claim. □

By using (7.1), we eliminate the curvature endomorphism $\mathfrak{R}^1_{(2_b,1_{a-b})}$ from (6.5) and (6.6).

Corollary 7.2. *On $S^k(\mathbf{H})\hat{\otimes}\Lambda^{a,b}_0(\mathbf{E})$, there is a Bochner-Weitzenböck formula depending only on the scalar curvature κ,*

$$
\begin{aligned}
&\sum_{N,\nu} (w_\nu + 2)\left(\pi_{(2_b,1_{a-b})}(c_2) + 4w_\nu^2 - 8nw_\nu - 12w_\nu\right) B_{N,\nu} \\
=& \frac{\kappa}{8n(n+2)} \pi_{(2_b,1_{a-b})}\left(-4(2n^2 + 7n + 7)c_2 + c_2^2 + 4c_4\right).
\end{aligned}
\tag{7.2}
$$

This formula is linear independent of (6.7)–(6.9).

8 Eigenvalue estimates

We shall apply our Bochner-Weitzenböck formulas to eigenvalue estimate on $\mathbf{S}_{k,(2_b,1_{a-b})} = S^k(\mathbf{H})\hat{\otimes}\Lambda^{a,b}_0(\mathbf{E})$ for $0 \le k \le 2n - a - b$ and $0 \le b \le a \le n$. This bundle is an irreducible summand of the bundle of differential forms [16]. We will discuss four cases in turn: (1) $a = b = 0$, (2) $a > b = 0$, (3) $a = b > 0$, (4) $a > b > 0$. In conclusion, the eigenvalue estimate of the Laplace operator $dd^* + d^*d$ restricted on $\mathbf{S}_{k,(2_b,1_{a-b})}$ is given in Theorem 8.5.

8.1 Estimates on $S^k(\mathbf{H})$

We consider the bundle $\mathbf{S}_{k,(0_n)} = S^k(\mathbf{H})$. There are two gradients, $D_{1,1}$ and $D_{-1,1}$. These operators satisfy

$$
B_{1,1} + B_{-1,1} = \nabla^*\nabla, \quad -kB_{1,1} + (k+2)B_{-1,1} = \frac{k(k+2)}{4(n+2)}\kappa,
$$

where $B_{\pm 1,1} = (D_{\pm 1,1})^* D_{\pm 1,1}$. Then we have

$$\nabla^*\nabla = \frac{2(k+1)}{k+2} B_{1,1} + \frac{k}{4(n+2)}\kappa = \frac{2(k+1)}{k} B_{-1,1} - \frac{k+2}{4(n+2)}\kappa.$$

We think of $S^k(\mathbf{H})$ as an irreducible summand of the bundle of differential forms. The restricted Laplacian $dd^* + d^*d$ on $S^k(\mathbf{H})$ is

$$\nabla^*\nabla + R^1_{k,(0_n)}/2 = \nabla^*\nabla + \frac{k(k+2)}{8n(n+2)}\kappa.$$

Then we have

$$\begin{aligned} dd^* + d^*d &= \frac{2(k+1)}{k+2} B_{1,1} + \frac{k(2n+k+2)}{8n(n+2)}\kappa \\ &= \frac{2(k+1)}{k} B_{-1,1} - \frac{(k+2)(2n-k)}{8n(n+2)}\kappa. \end{aligned}$$

This equation leads to eigenvalue estimates of $dd^* + d^*d$ (cf. [17]).

Proposition 8.1. *We consider a compact quaternionic Kähler manifold with non-zero scalar curvature. A lower bound of the eigenvalues of $dd^* + d^*d$ on $S^k(\mathbf{H})$ for non-negative integer k is given as follows.*

$$\begin{cases} \dfrac{k(2n+k+2)}{8n(n+2)}\kappa & \textit{for } \kappa > 0, \\ -\dfrac{(k+2)(2n-k)}{8n(n+2)}\kappa & \textit{for } \kappa < 0. \end{cases}$$

8.2 Estimates on $S^k(\mathbf{H})\hat{\otimes}\Lambda^a_0(\mathbf{E})$

We consider $\mathbf{S}_{k,(1_a)} = S^k(\mathbf{H})\hat{\otimes}\Lambda^a_0(\mathbf{E})$ for $a \neq 0$. Then we have six gradients on this vector bundle, $D_{1,1}$, $D_{1,a+1}$, $D_{1,-a}$, $D_{-1,1}$, $D_{-1,a+1}$ and $D_{-1,-a}$, where we set $D_{\pm 1,a+1} := 0$ in the case of $a = n$.

The sum of $B_{N,\nu} = (D_{N,\nu})^* D_{N,\nu}$ is the connection Laplacian,

$$B_{1,1} + B_{1,a+1} + B_{1,-a} + B_{-1,1} + B_{-1,a+1} + B_{-1,-a} = \nabla^*\nabla. \tag{8.1}$$

Moreover, it follows from (6.5), (6.7) and (6.8) that there are three independent Bochner-Weitzenböck formulas,

$$\begin{aligned}
&- B_{1,1} + aB_{1,a+1} + (2n-a+2)B_{1,-a} \\
&\quad - B_{-1,1} + aB_{-1,a+1} + (2n-a+2)B_{-1,-a} = \frac{a(2n-a+2)}{4n(n+2)}\kappa, \\
&\quad\quad - k(B_{1,1} + B_{1,a+1} + B_{1,-a}) \\
&\quad\quad\quad + (k+2)(B_{-1,1} + B_{-1,a+1} + B_{-1,-a}) = \frac{k(k+2)}{4(n+2)}\kappa, \\
&- k(n+2)B_{1,1} + ka(n-a+1)B_{1,a+1} - k(2n-a+2)(n-a+1)B_{1,-a} \\
&+ (k+2)(n+2)B_{-1,1} - (k+2)a(n-a+1)B_{-1,a+1} \\
&+ (k+2)(2n-a+2)(n-a+1)B_{-1,-a} = \frac{k(k+2)a(2n-a+2)}{4n(n+2)}\kappa.
\end{aligned} \tag{8.2}$$

From these formulas, we change the connection Laplacian $\nabla^*\nabla$ into the form of

$$\sum_{N,\nu} c_{N,\nu} B_{N,\nu} + c\kappa, \qquad \text{for } c_{N,\nu} \geq 0. \tag{8.3}$$

Since $B_{N,\nu}$ is non-negative operator on a compact quaternionic Kähler manifold, the eigenvalues of $\nabla^*\nabla$ have a lower bound $c\kappa$. For example, it follows from (8.1) and (8.2) that

$$\nabla^*\nabla = \frac{2(k+1)}{k+2}(B_{1,1} + B_{1,a+1} + B_{1,-a}) + \frac{k}{4(n+2)}\kappa.$$

Then the eigenvalues of $\nabla^*\nabla$ on a compact positive quaternionic Kähler manifold have a lower bound $\frac{k}{4(n+2)}\kappa$. Thus, to get a lower bound of the eigenvalues of $\nabla^*\nabla$, we should find out a formula of (8.3) such that $c\kappa$ is as great as possible. In fact we can rewrite $\nabla^*\nabla$ as follows.

1. In the case that the scalar curvature κ is positive,

   (a) For $k = 0$,

   $$\nabla^*\nabla = \frac{(2n-a+3)}{(2n-a+2)}B_{1,1} + \frac{2(n-a+1)}{(2n-a+2)}B_{1,a+1} + \frac{a}{4n(n+2)}\kappa.$$

   (b) For $k \neq 0$,

   $$\nabla^*\nabla = \frac{2(k+1)}{k+2}(B_{1,1} + B_{1,a+1} + B_{1,-a}) + \frac{k}{4(n+2)}\kappa.$$

2. In the case that the scalar curvature κ is negative,

   (a) For $0 \le k \le n-a$,

$$\nabla^*\nabla = \frac{2(a+1)(n-k-a)}{(k+2)(n-a)}B_{1,a+1} + \frac{2(k+1)(2n-a+3)}{k+2}B_{1,-a} + \frac{2(a+1)(n-a+1)}{n-a}B_{-1,a+1} - \frac{2an+kn-a^2-ka+2a+2k}{4n(n+2)}\kappa.$$

   (b) For $n-a < k \le 2n-a$,

$$\nabla^*\nabla = \frac{2(2n-a+3)(n-a+1)}{n-a+2}B_{1,-a} + \frac{2(k+1)(a+1)}{k}B_{-1,a+1} + \frac{2(2n-a+3)(k+a-n)}{k(n-a+2)}B_{-1,-a} - \frac{-ka-a^2+2n+kn+2an}{4n(n+2)}\kappa.$$

Then we have an eigenvalue estimate of $\nabla^*\nabla$ on $\mathbf{S}_{k,(1_a)}$.

Proposition 8.2. *We consider the connection Laplacian* $\nabla^*\nabla$ *on* $\mathbf{S}_{k,(1_a)} = S^k(\mathbf{H})\hat{\otimes}\Lambda^a_0(\mathbf{E})$. *A lower bound of the eigenvalues of* $\nabla^*\nabla$ *is as follows.*

1. *On a compact positive quaternionic Kähler manifold,*

$$\begin{cases} \dfrac{a}{4n(n+2)}\kappa & \text{for } k=0, \\ \dfrac{k}{4(n+2)}\kappa & \text{for } k\neq 0. \end{cases}$$

2. *On a compact negative quaternionic Kähler manifold,*

$$\begin{cases} -\dfrac{2an+kn-a^2-ka+2a+2k}{4n(n+2)}\kappa & \text{for } 0\le k\le n-a, \\ -\dfrac{-ka-a^2+2n+kn+2an}{4n(n+2)}\kappa & \text{for } n-a<k\le 2n-a. \end{cases}$$

Example 8.1 (The Dirac operator [10]). We consider the Dirac operator on the spinor bundle $\bigoplus_{k=0}^n \mathbf{S}_{k,(1_{n-k})}$. Because of $D^2 = \nabla^*\nabla + \kappa/4$, the eigenvalues of D^2 on a compact positive quaternionic Kähler spin manifold have the following lower bound,

$$\begin{cases} \dfrac{n+3}{4(n+2)}\kappa & \text{for } k=0, \\ \dfrac{n+k+2}{4(n+2)}\kappa & \text{for } 0<k\le n. \end{cases}$$

Example 8.2 (The Laplacian). We think of $\mathbf{S}_{k,(1_a)}$ as an irreducible summand of the bundle of differential forms. The restricted Laplacian $dd^* + d^*d$ on $\mathbf{S}_{k,(1_a)}$ is

$$\nabla^*\nabla + R^1_{k,(1_a)}/2 = \nabla^*\nabla + \frac{\kappa}{8n(n+2)}(k(k+2) + a(2n-a+2)).$$

Then we have eigenvalue estimates of $dd^* + d^*d$ on a compact quaternionic Kähler manifold with non-zero scalar curvature.

1. When $\kappa > 0$, a lower bound of the eigenvalues of $dd^* + d^*d$ is

$$\begin{cases} \dfrac{a(2n-a+4)}{8n(n+2)}\kappa & \text{for } k = 0, \\ \dfrac{(a+k)(2n-a+k+2)}{8n(n+2)}\kappa & \text{for } k \neq 0. \end{cases}$$

2. When $\kappa < 0$, a lower bound is

$$\begin{cases} -\dfrac{(a+k)(2n-a-k+2)}{8n(n+2)}\kappa & \text{for } 0 \leq k \leq n-a, \\ -\dfrac{(a+k+2)(2n-a-k)}{8n(n+2)}\kappa & \text{for } n-a < k \leq 2n-a. \end{cases}$$

These results give eigenvalue estimates of $dd^* + d^*d$ on $\Lambda^1(M) = \mathbf{H}\hat{\otimes}\mathbf{E}$ in [1], [13], [16]. Next we consider the bundle of 2-forms,

$$\Lambda^2(M) = \mathbf{S}_{2,(0_n)} \oplus \mathbf{S}_{2,(1_2,0_{n-2})} \oplus \mathbf{S}_{0,(2,0_{n-1})}.$$

A lower bound of the eigenvalues of $dd^* + d^*d$ on $\mathbf{S}_{2,(1_2,0_{n-2})}$ is

$$\begin{cases} \dfrac{n+1}{n(n+2)}\kappa & \text{for } \kappa > 0, \\ -\dfrac{n-1}{n(n+2)}\kappa & \text{for } \kappa < 0 \text{ and } n \geq 4, \\ -\dfrac{3(n-2)}{2n(n+2)}\kappa & \text{for } \kappa < 0 \text{ and } n = 2, 3. \end{cases}$$

We know that our estimates for $\kappa < 0$ are better than the ones in [16].

Now, we shall apply Bochner-Weitzenböck formulas to vanishing theorems. We consider the vector bundle $S^{k+1}(\mathbf{H})\hat{\otimes}\mathbf{E}$ on a compact quaternionic Kähler manifold M. It follows from the Penrose transform that the Dolbeault cohomology $H^1(Z, \mathcal{O}(k))$ $(k \geq 0)$ on the twistor space Z of M is isomorphic to the solution space of certain linear differential equation on M(see [9], [11]). We can easily show that the solution space $\mathcal{S}$ is given by

$$\mathcal{S} = \ker D_{1,2} \cap \ker D_{1,-1} \cap \ker D_{-1,-1} \subset \Gamma(M, S^{k+1}(\mathbf{H})\hat{\otimes}\mathbf{E}).$$

Because of (8.2), a solution ϕ in $\mathcal{S}$ satisfies

$$\|D_{1,1}\phi\|^2 = \int_M \langle D_{1,1}\phi, D_{1,1}\phi\rangle \mathrm{vol}_g = -\frac{(k+3)(2n+k+2)}{8n(n+2)(k+2)}\kappa\|\phi\|^2, \tag{8.4}$$

$$\|D_{-1,1}\phi\|^2 = \frac{k(k+1)}{8(n+2)(k+2)}\kappa\|\phi\|^2,$$

$$\|D_{-1,2}\phi\|^2 = \frac{(k+1)(n-1)}{8n(n+2)}\kappa\|\phi\|^2. \tag{8.5}$$

If the scalar curvature is negative, then the equation (8.5) yields $\mathcal{S} = \{0\}$. This vanishing was shown in [9]. When the scalar curvature is positive, we also have $\mathcal{S} = \{0\}$ by (8.4) (cf. [12]).

Proposition 8.3. *Let Z be the twistor space of a compact quaternionic Kähler manifold with non-zero scalar curvature. Then we have $H^1(Z, \mathcal{O}(k)) = 0$ for non-negative integer k.*

The author expect that our Bochner-Weitzenböck formulas will produce vanishing theorems for higher cohomology $H^i(Z, \mathcal{O}(k))$, for example, vanishing theorems in [11].

8.3 Estimates on $S^k(\mathbf{H})\hat{\otimes}\Lambda_0^{a,a}(\mathbf{E})$

We consider the vector bundle $\mathbf{S}_{k,(2_a)} = S^k(\mathbf{H})\hat{\otimes}\Lambda_0^{a,a}(\mathbf{E})$ as an irreducible summand of the bundle of differential forms, where we assume that a is not zero. On this vector bundle, we have six gradients, $D_{\pm1,1}$, $D_{\pm1,a+1}$ and $D_{\pm1,-a}$. The Laplace operator $dd^* + d^*d$ restricted to $\mathbf{S}_{k,(2_a)}$ is

$$\nabla^*\nabla + R^1_{k,(2_a)}/2 = \sum_{N,\nu}\left(1 + \frac{w_\nu}{2} + \frac{W_N}{2n}\right)B_{N,\nu}$$

$$= -\frac{k}{2n}B_{1,1} + \frac{2n+an-k}{2n}B_{1,a+1} + \frac{2n^2+5n-an-k}{2n}B_{1,-a}$$
$$+\frac{k+2}{2n}B_{-1,1} + \frac{2n+an+k+2}{2n}B_{-1,a+1} + \frac{2n^2+5n-an+k+2}{2n}B_{-1,-a}.$$

There are three independent Bochner-Weitzenböck formulas (6.5), (6.7) and (6.8). Since (6.5) includes $\mathfrak{R}^1_{k,(2_a)}$, we use only (6.7) and (6.8) to estimate eigenvalues. We rewrite dd^*+d^*d to the form of (8.3) such that $c\kappa$ is greatest.

1. In the case of $\kappa > 0$,

$$\begin{aligned} & dd^* + d^*d \\ =& \frac{(k+1)(a+2)}{k+2} B_{1,a+1} + \frac{(2n-a+5)(2n-2a+3k+6)}{2(k+2)(n-a+3)} B_{1,-a} \\ & + \frac{(2n-a+5)(2n-2a+3)}{2(n-a+3)} B_{-1,-a} + \frac{k(2n-2a+k+2)}{8n(n+2)} \kappa. \end{aligned}$$

2. In the case of $\kappa < 0$,

   (a) For $0 \leq k \leq \frac{2n-2a}{3}$,

$$\begin{aligned} & dd^* + d^*d \\ =& \frac{(a+2)(2n-2a-3k)}{2(k+2)(n-a)} B_{1,a+1} + \frac{(k+1)(2n-a+5)}{k+2} B_{1,-a} \\ & + \frac{(a+2)(2n-2a+3)}{2(n-a)} B_{-1,a+1} - \frac{k(2n-2a-k+4)}{8n(n+2)} \kappa. \end{aligned}$$

   (b) For $\frac{2n-2a}{3} < k \leq 2n-2a$,

$$\begin{aligned} & dd^* + d^*d \\ =& \frac{(2n-a+5)(2n-2a+3)}{2(n-a+3)} B_{1,-a} + \frac{(k+1)(a+2)}{k} B_{-1,a+1} \\ & + \frac{(2n-a+5)(3k+2a-2n)}{2k(n-a+3)} B_{-1,-a} - \frac{(k+2)(2n-2a-k)}{8n(n+2)} \kappa. \end{aligned}$$

Proposition 8.4. *A lower bound of the eigenvalues of $dd^* + d^*d$ on $\mathbf{S}_{k,(2_a)}$ is as follows.*

1. *On a compact positive quaternionic Kähler manifold,*

$$\frac{k(2n-2a+k+2)}{8n(n+2)} \kappa.$$

2. *On a compact negative quaternionic Kähler manifold,*

$$\begin{cases} -\dfrac{k(2n-2a-k+4)}{8n(n+2)} \kappa & \text{for } 0 \leq k \leq \frac{2n-2a}{3}, \\ -\dfrac{(k+2)(2n-2a-k)}{8n(n+2)} \kappa & \text{for } \frac{2n-2a}{3} < k \leq 2n-2a. \end{cases}$$

8.4 Estimates on $S^k(\mathbf{H})\hat{\otimes}\Lambda_0^{a,b}(\mathbf{E})$

We consider $\mathbf{S}_{k,(2_b,1_{a-b})} = S^k(\mathbf{H})\hat{\otimes}\Lambda_0^{a,b}(\mathbf{E})$ for $a > b > 0$. We have ten gradients on this bundle,

$$D_{\pm 1,1}, \quad D_{\pm 1,b+1}, \quad D_{\pm 1,a+1}, \quad D_{\pm 1,-b}, \quad D_{\pm 1,-a}.$$

The Laplacian $dd^* + dd^*$ is

$$dd^* + d^*d = \nabla^*\nabla + R^1_{k,(2_b,1_{a-b})}/2 = \sum_{N,\nu}\left(1 + \frac{w_\nu}{2} + \frac{W_N}{2n}\right) B_{N,\nu},$$

and there are four Bochner-Weitzenböck formulas (6.7)–(6.9) and (7.2) depending only on the scalar curvature. By a tedious calculation, we can rewrite $dd^* + d^*d$ as follows:
On a positive quaternionic Kähler manifold,

1. For $k = 0$,

$$\begin{aligned} &dd^* + d^*d \\ =&\frac{(b+1)(2n-a-b+3)}{2n-a-b+2}B_{1,b+1} + \frac{2(a+2)(n-a+1)}{(a-b+2)(2n-a-b+2)}B_{1,a+1} \\ &+ \frac{(a-b+1)(2n-b+5)}{a-b+2}B_{1,-b} + \frac{(a-b)(2n-a-b+4)}{8n(n+2)}\kappa. \end{aligned}$$

2. For $k \neq 0$,

$$\begin{aligned} &dd^* + d^*d \\ =&\frac{2(b+1)(k+1)}{k+2}B_{1,b+1} + \frac{2(a+2)(k+1)}{(k+2)(a-b+2)}B_{1,a+1} \\ &+ \frac{2(2n-b+5)}{(a-b+2)(k+2)(2n-a-b+4)(n-b+3)}\times \\ &(12 + 6a - 3a^2 - 16b - 2ab + a^2b + 7b^2 - b^3 + 6k + 2ak - a^2k \\ &- 5bk + b^2k + 10n + 8an - a^2n - 12bn - 2abn + 3b^2n \\ &+ 3kn + 2akn - 2bkn + 2n^2 + 2an^2 - 2bn^2)B_{1,-b} \\ &+ \frac{2(k+1)(2n-a+4)}{(k+2)(2n-a-b+4)}B_{1,-a} \\ &+ \frac{2(a-b+1)(2n-b+5)(2n-a-b+3)(n-b+2)}{(a-b+2)(2n-a-b+4)(n-b+3)}B_{-1,-b} \\ &+ \frac{(a-b+k)(2n-a-b+k+2)}{8n(n+2)}\kappa. \end{aligned}$$

Here, substituting $t = a - b \geq 0$ and $s = n - a \geq 0$, we can verify that the coefficient of $B_{1,-b}$ is non-negative.

On a negative quaternionic Kähler manifold,

1. For $0 \leq k \leq n - a$,

$$
\begin{aligned}
& dd^* + d^*d \\
=& \frac{2(a+2)(a-b+1)(n-a-k)}{(a-b+2)(k+2)(n-a)} B_{1,a+1} \\
&+ \frac{2(2n-b+5)(n-b+2)(2kn-ak-bk+2n-2b+5k+6)}{(a-b+2)(k+2)(2n-a-b+4)(n-b+3)} B_{1,-b} \\
&+ \frac{2(k+1)(2n-a+4)(2n-a-b+3)}{(k+2)(2n-a-b+4)} B_{1,-a} \\
&+ \frac{2(a+2)(a-b+1)(n-a+1)}{(a-b+2)(n-a)} B_{-1,a+1} \\
&+ \frac{2(a-b+1)(2n-b+5)(n-b+2)}{(a-b+2)(2n-a-b+4)(n-b+3)} B_{-1,-b} \\
&- \frac{(a-b+k)(2n-a-b-k+2)}{8n(n+2)} \kappa.
\end{aligned}
$$

2. For $n - a < k \leq 2n - a - b$,

$$
\begin{aligned}
& dd^* + d^*d \\
=& \frac{2(2n-b+5)(2n-a-b+3)(n-b+2)}{(a-b+2)(2n-a-b+4)(n-b+3)} B_{1,-b} \\
&+ \frac{2(2n-a+4)(2n-a-b+3)(n-a+1)}{(n-a+2)(2n-a-b+4)} B_{1,-a} \\
&+ \frac{2(a+2)(a-b+1)(k+1)}{k(a-b+2)} B_{-1,a+1} \\
&+ \frac{2(2n-b+5)(n-b+2)(2a+3k+ak-bk-2n)}{(a-b+2)k(2n-a-b+4)(n-b+3)} B_{-1,-b} \\
&+ \frac{2(2n-a+4)(2n-a-b+3)(a+k-n)}{k(2n-a-b+4)(n-a+2)} B_{-1,-a} \\
&- \frac{(a-b+k+2)(2n-a-b-k)}{8n(n+2)} \kappa.
\end{aligned}
$$

Then we have a lower bound of the eigenvalues of $dd^* + d^*d$ on $\mathbf{S}_{k,(2_b,1_{a-b})}$ for $a > b > 0$.

From the results given in this section, we complete an eigenvalue estimate of $dd^* + dd^*$ on $S^k(\mathbf{H})\hat{\otimes}\Lambda_0^{a,b}(\mathbf{E})$.

Theorem 8.5. *The eigenvalues of the Laplace operator* dd^*+d^*d *on* $S^k(\mathbf{H})\hat{\otimes}\Lambda_0^{a,b}(\mathbf{E})$ *for* $0 \le k \le 2n-a-b$ *and* $0 \le b \le a \le n$ *have the following lower bound.*

1. *On a compact positive quaternionic Kähler manifold,*

$$\begin{cases} \dfrac{(a-b)(2n-a-b+4)}{8n(n+2)}\kappa & \text{for } k=0, \\ \dfrac{(a-b+k)(2n-a-b+k+2)}{8n(n+2)}\kappa & \text{for } k \neq 0. \end{cases}$$

2. *On a compact negative quaternionic Kähler manifold,*

   (a) *when* $a=b=0$,

$$-\frac{(k+2)(2n-k)}{8n(n+2)}\kappa.$$

   (b) *when* $a=b>0$,

$$\begin{cases} -\dfrac{k(2n-2a-k+4)}{8n(n+2)}\kappa & \text{for } 0 \le k \le \frac{2n-2a}{3}, \\ -\dfrac{(k+2)(2n-2a-k)}{8n(n+2)}\kappa & \text{for } \frac{2n-2a}{3} < k \le 2n-2a. \end{cases}$$

   (c) *when* $a>b\ge 0$,

$$\begin{cases} -\dfrac{(a-b+k)(2n-a-b-k+2)}{8n(n+2)}\kappa & \text{for } 0 \le k \le n-a, \\ -\dfrac{(a-b+k+2)(2n-a-b-k)}{8n(n+2)}\kappa & \text{for } n-a < k \le 2n-a-b. \end{cases}$$

From this theorem, we know which irreducible bundles carry harmonic forms. The next corollary leads to the weak Lefschetz theorems for quaternionic Kähler manifolds in [15] and [16].

Corollary 8.6 ([16])**.** *We consider the bundle of differential forms on a compact quaternionic Kähler manifold. If the scalar curvature is positive, a harmonic form is a section of* $\Lambda_0^{a,a}(\mathbf{E})$ *for* $0 \le a \le n$. *If the scalar curvature is negative, a harmonic form is a section of* $\Lambda_0^{a,a}(\mathbf{E})$ *for* $0 \le a \le n$, *or* $S^{2n-a-b}(\mathbf{H})\hat{\otimes}\Lambda_0^{a,b}(\mathbf{E})$ *for* $0 \le b \le a \le n$.

We shall finish by discussing relations between our eigenvalue estimates and the first eigenvalues on the quaternionic projective space $\mathbb{H}P^n$ with $\kappa = 2n$. In [18], C. Tsukamoto calculated the spectra of the Laplace operator $dd^* + d^*d$ on $\mathbb{H}P^n$. On $S^k(\mathbf{H})\hat{\otimes}\Lambda^a_0(\mathbf{E})$, the first eigenvalue coincides with the lower bound in Theorem 8.5. But, so does not on $S^k(\mathbf{H})\hat{\otimes}\Lambda^{a,b}_0(\mathbf{E})$ for $a \geq b > 0$. The reason is that we use only (6.7)–(6.9) and (7.2). Since the curvature R^{hyper} is zero on $\mathbb{H}P^n$, we can use all Bochner-Weitzenböck formulas (6.5)–(6.9) to estimate eigenvalues. Then we have a better eigenvalue estimate which coincides with the first eigenvalue on $\mathbb{H}P^n$.

Example 8.3. We consider the Laplace operator $dd^* + d^*d$ on $S^k(\mathbf{H})\hat{\otimes}\Lambda^{a,b}_0(\mathbf{E})$. We can easily show that, for $k \geq 2$, the first eigenvalue λ_1 on $\mathbb{H}P^n$ is

$$\frac{1}{4(n+2)}(k(k+2n+2) + a(2n-a+2) + b(2n-b+4)).$$

On the other hand, it follows from (6.5) and (6.7) that

$$\begin{aligned} dd^* + d^*d =& \sum_{N,\nu} \left(1 + \frac{w_\nu}{2} + \frac{W_N}{2n}\right) B_{N,\nu} \\ =& \frac{1}{8(n+2)}(2k(k+2) + \pi_{(2_b,1_{a-b})}(c_2)) + \sum_{N,\nu} B_{N,\nu} \\ =& \frac{1}{8(n+2)}(2k(k+2n+2) + \pi_{(2_b,1_{a-b})}(c_2)) + \sum_{\nu} \frac{2(k+1)}{k+2} B_{1,\nu} \\ =& \lambda_1 + \sum_{\nu} \frac{2(k+1)}{k+2} B_{1,\nu}. \end{aligned}$$

Thus we verify that the lower bound induced from Bochner-Weitzenböck formulas coincides with the first eigenvalue on $\mathbb{H}P^n$.

Acknowledgement

The author was partially supported by the Grant-in-Aid for JSPS Research Fellowships for Young Scientists.

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