Universal Bochner-Weitzenböck formulas for hyper-Kählerian gradients

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Abstract. Hyper-Kählerian gradients on hyper-Kähler manifolds are first-order differential operators naturally defined by hyper-Kähler structure. We show that the principal symbols of hyper-Kählerian gradients are related to the enveloping algebra and Casimir elements of the symplectic group. In particular, we give universal Bochner-Weitzenböck formulas which are certain relations in the enveloping algebra. From the formulas, we construct Bochner-Weitzenböck formulas for hyper-Kählerian gradients.

1. Introduction

Let (M, g, I, J, K) be a real $4n$-dimensional hyper-Kähler manifold. The hyper-Kähler structure gives a reduction of the frame bundle with the Levi-Civita connection. We denote the principal $Sp(n)$-bundle on M by $\mathbf{Sp}(M)$. Considering an irreducible unitary representation (π_ρ, V_ρ) of $Sp(n)$ with highest weight ρ, we have a Hermitian vector bundle $\mathbf{S}_\rho := \mathbf{Sp}(M) \times_\rho V_\rho$. The Levi-Civita connection induces a covariant derivative ∇ on $\mathbf{S}_\rho$,

$$\nabla : \Gamma(\mathbf{S}_\rho) \xrightarrow{\nabla} \Gamma(\mathbf{S}_\rho \otimes (\Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M))).$$

Here, $\Lambda^{1,0}(M)$ and $\Lambda^{0,1}(M)$ are the $(1,0)$- and the $(0,1)$-cotangent bundles with respect to the complex structure I. Since both $\mathbf{S}_\rho$ and $\Lambda^{1,0}(M)$ are associated bundles to $\mathbf{Sp}(M)$, we decompose the tensor bundle $\mathbf{S}_\rho \otimes \Lambda^{1,0}(M)$ into the sum of irreducible bundles, $\mathbf{S}_\rho \otimes \Lambda^{1,0}(M) = \oplus_\nu \mathbf{S}_\nu$. Then we have a first-order differential operator as a component of the $(1,0)$-covariant derivative $\nabla^{1,0}$,

$$D_\nu : \Gamma(\mathbf{S}_\rho) \xrightarrow{\nabla^{1,0}} \Gamma(\mathbf{S}_\rho \otimes \Lambda^{1,0}(M)) \xrightarrow{\Pi_\nu} \Gamma(\mathbf{S}_\nu),$$

where Π_ν is the orthogonal projection from $\mathbf{S}_\rho \otimes \Lambda^{1,0}(M)$ onto $\mathbf{S}_\nu$. Since there is a real or quaternionic structure $\mathfrak{J}$ on each associated vector bundle, we twist D_ν by $\mathfrak{J}$ and have another first-order differential operator $\mathfrak{J} D_\nu \mathfrak{J}^{-1} : \Gamma(\mathbf{S}_\rho) \to \Gamma(\mathbf{S}_\nu)$ (see

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Section 4). In fact, we know that $\mathfrak{J} D_\nu \mathfrak{J}^{-1}$ is a component of $\nabla^{0,1}$. We call these operators $\{D_\nu, \mathfrak{J} D_\nu \mathfrak{J}^{-1}\}_\nu$ *hyper-Kählerian gradients* on M.

A usual gradient is a first-order differential operator as a component of covariant derivative on a Riemannian or spin manifold. In recent research for gradients by Thomas Branson et al., we know that gradients have importance and many applications in geometry and analysis (see [1], [2], [3], [4], [6], [8], and a series of papers about gradients by T. Branson). In fact, many geometric first-order differential operators are gradients: the Dirac operator, the twistor operator, the exterior derivative, the interior derivative, the conformal Killing operator, the Rarita-Schwinger operator, and so on. Since gradients are defined through fiber-wise orthogonal projections, their principal symbols are related to representation theory for the special orthogonal group $SO(n)$ or the spin group $Spin(n)$ [1], [4], [8]. It is natural to change the group of symmetry, $SO(n)$, to other Lie groups. For the case of the unitary group $U(n)$, the author has discussed Kählerian gradients on Kähler manifolds and given some applications to Kähler geometry [7]. In this paper, we discuss the case of the symplectic group $Sp(n)$, that is, hyper-Kählerian gradients on hyper-Kähler manifolds.

Our main purpose is to construct Bochner-Weitzenböck formulas for hyper-Kählerian gradients. From the definition of them, we can show that

$$\sum_\nu D_\nu^* D_\nu = \nabla^{1,0*}\nabla^{1,0}, \quad \sum_\nu \mathfrak{J} D_\nu^* D_\nu \mathfrak{J}^{-1} = \nabla^{0,1*}\nabla^{0,1}. \tag{1.1}$$

Then our task is to find out independent vectors $(\{a_\nu\}_\nu, \{b_\nu\}_\nu, \{c_\nu\}_\nu, \{d_\nu\}_\nu)$ as many as possible such that

$$\sum_\nu a_\nu D_\nu^* D_\nu + b_\nu \mathfrak{J} D_\nu^* D_\nu \mathfrak{J}^{-1} + c_\nu \mathfrak{J} D_\nu^* \mathfrak{J}^{-1} D_\nu + d_\nu D_\nu^* \mathfrak{J} D_\nu \mathfrak{J}^{-1} = \text{curvature action}. \tag{1.2}$$

We call these identities (1.1) and (1.2) *Bochner-Weitzenböck formulas* or *Bochner identities*. These formulas are useful to make vanishing theorems and eigenvalue estimates [2], [7], [10], [11]. The equation (1.2) means that the principal symbol of the operator on the left vanishes. In other words, there are certain relations among principal symbols of hyper-Kählerian gradients such as the Clifford relation. Our plan to construct Bochner-Weitzenböck formulas is as follows:

1. We relate the principal symbols to the (universal) enveloping algebra $U(\mathfrak{sp}(n,\mathbb{C}))$ of $Sp(n)$.
2. We find out certain relations on $U(\mathfrak{sp}(n,\mathbb{C}))$ through study of Casimir elements.
3. The relations give desired algebraic relations for the principal symbols. Furthermore, they also give some identities for Casimir elements.

As a result, we know that there is a correspondence between identities for Casimir elements of $Sp(n)$ and Bochner-Weitzenböck formulas for hyper-Kählerian gradients.

This paper is organized as follows: In Section 2, we give a review to representation theory for symplectic groups. In Section 3, we introduce Clifford homomorphisms on $Sp(n)$-modules as a generalization of Clifford multiplication on spinor spaces. We relate them to the enveloping algebra of $Sp(n)$. Through study of Casimir elements, we find out the universal Bochner-Weitzenböck formulas, which induce not only identities for Casimir elements but also relations for Clifford homomorphisms. In Section 4, we define hyper-Hermitian gradients on hyper-Hermitian manifolds and prove their conformal covariance. In the last section, by using the universal Bochner-Weitzenböck formulas, we give Bochner-Weitzenböck formulas for hyper-Kählerian gradients on hyper-Kähler manifolds.

2. Representation theory for symplectic groups

Let W be a real $4n$-dimensional vector space with an inner product g. We call (I, J, K) a *hyper-Hermitian structure* on (W, g) if I, J and K are complex structures on W such that $IJ = -JI = K$ and $g(Ix, Iy) = g(Jx, Jy) = g(x, y)$ for any x, y in W. In other words, W is a quaternionic vector space with a compatible inner product g.

We consider the complexification of W, $W \otimes \mathbb{C}$, and extend I, J, K and g complex linearly on $W \otimes \mathbb{C}$. The complex vector space $W \otimes \mathbb{C}$ splits into the direct sum of $W^{1,0}$ and $W^{0,1}$ with respect to I. These vector spaces have Hermitian inner products given by $(u, v) := g(u, \bar{v})$, where $\bar{v}$ is complex conjugate of v. Another complex structure J gives a complex-linear map $J : W^{1,0} \to W^{0,1}$ because of $JI = -IJ$. Then $W^{1,0}$ has an anti-linear map $\mathfrak{J} : W^{1,0} \ni u \mapsto J(\bar{u}) \in W^{1,0}$ such that $\mathfrak{J}^2 = -1$ and $(\mathfrak{J}u, \mathfrak{J}v) = (v, u)$. This means that $W^{1,0}$ has a quaternionic structure $\mathfrak{J}$ compatible with Hermitian inner product. The space $W^{0,1}$ also has such a structure. Moreover $W^{1,0}$ and $W^{0,1}$ have complex symplectic structures. In fact,

$$\Omega^{2,0}(u, v) := (g(Ju, v) + \sqrt{-1}g(Ku, v))/2, \quad \Omega^{0,2} := (g(Ju, v) - \sqrt{-1}g(Ku, v))/2$$

are complex symplectic forms on $W^{1,0}$ and $W^{0,1}$, respectively.

On a hyper-Hermitian vector space (W, g, I, J, K), we choose an orthonormal basis $\{e_k, Ie_k, Je_k, Ke_k\}_{k=1}^n$ and set

$$\epsilon_k := \frac{1}{\sqrt{2}}(e_k - \sqrt{-1}Ie_k), \quad \epsilon_{-k} := \frac{1}{\sqrt{2}}(Je_k - \sqrt{-1}IJe_k),$$
$$\bar{\epsilon}_k := \frac{1}{\sqrt{2}}(e_k + \sqrt{-1}Ie_k), \quad \bar{\epsilon}_{-k} := \frac{1}{\sqrt{2}}(Je_k + \sqrt{-1}IJe_k),$$

for $k = 1, \cdots, n$. These vectors constitute unitary bases $\{\epsilon_\alpha\}_{\alpha=\pm1}^{\pm n}$ of $W^{1,0}$ and $\{\bar{\epsilon}_\alpha\}_{\alpha=\pm1}^{\pm n}$ of $W^{0,1}$ such that

$$\mathfrak{J}(\epsilon_\alpha) = \mathrm{sign}(\alpha)\epsilon_{-\alpha}, \quad \mathfrak{J}(\bar{\epsilon}_\alpha) = \mathrm{sign}(\alpha)\bar{\epsilon}_{-\alpha},$$
$$\Omega^{2,0}(\epsilon_\alpha, \epsilon_\beta) = \mathrm{sign}(\alpha)\delta_{-\alpha\beta}, \quad \Omega^{0,2}(\bar{\epsilon}_\alpha, \bar{\epsilon}_\beta) = \mathrm{sign}(\alpha)\delta_{-\alpha\beta},$$

where sign(α) is the sign of α,

$$\operatorname{sign}(\alpha) = \begin{cases} 1 & \text{for } \alpha > 0, \\ -1 & \text{for } \alpha < 0. \end{cases}$$

The symplectic group $Sp(n)$ is a group of automorphisms on W preserving hyper-Hermitian structure,

$$Sp(n) := \{A \in SO(W) \mid AI = IA, AJ = JA, KA = AK\}.$$

This group is also realized on $W^{1,0}$ as

$$\begin{aligned} Sp(n) &= \{A \in U(W^{1,0}) \mid \Omega^{2,0}(Au, Av) = \Omega^{2,0}(u, v) \text{ for } u \text{ and } v \text{ in } W^{1,0}\} \\ &= \{A \in U(W^{1,0}) \mid \mathfrak{J} A \mathfrak{J}^{-1} = A\}. \end{aligned}$$

The complexification $Sp(n, \mathbb{C})$ of $Sp(n)$ is called the complex symplectic group, which preserves $\Omega^{2,0}$ on $W^{1,0}$,

$$Sp(n, \mathbb{C}) = \{A \in GL_{\mathbb{C}}(W^{1,0}) \mid \Omega^{2,0}(Au, Av) = \Omega^{2,0}(u, v) \text{ for } u \text{ and } v \text{ in } W^{1,0}\}.$$

There are matrix representations of the Lie algebra $\mathfrak{sp}(n)$ and $\mathfrak{sp}(n, \mathbb{C})$ with our unitary basis $\{\epsilon_\alpha\}_\alpha$,

$$\mathfrak{sp}(n) = \{\begin{pmatrix} A & B \\ -B^* & \bar{A} \end{pmatrix} \mid A + A^* = 0, {}^tB = B\},$$

$$\mathfrak{sp}(n, \mathbb{C}) = \{\begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \mid {}^tC = C, {}^tB = B\}.$$

Then we define canonical elements of $\mathfrak{sp}(n, \mathbb{C})$ by

$$x_{\alpha\beta} := \epsilon_\alpha \otimes \epsilon_\beta^* - \operatorname{sign}(\alpha\beta)\epsilon_{-\beta} \otimes \epsilon_{-\alpha}^* \quad \text{in } W^{1,0} \otimes (W^{1,0})^* = \operatorname{End}(W^{1,0}) \tag{2.1}$$

for $\alpha, \beta = \pm 1, \cdots, \pm n$. Here, the basis of $\mathfrak{sp}(n, \mathbb{C})$ is $\{x_{\alpha\beta} | \alpha + \beta \geq 0\}$.

Lemma 2.1. 1. *The Lie algebra* $\mathfrak{sp}(n, \mathbb{C})$ *is spanned by* $\{x_{\alpha\beta}\}_{\alpha,\beta=\pm1}^{\pm n}$ *with relations*

$$\begin{aligned} x_{\alpha\beta} &= -\operatorname{sign}(\alpha\beta) x_{-\beta -\alpha}, \\ [x_{\alpha\beta}, x_{\mu\nu}] &= \delta_{\beta\mu} x_{\alpha\nu} - \delta_{\alpha\nu} x_{\mu\beta} + \operatorname{sign}(\alpha\beta)(\delta_{-\beta\nu} x_{\mu -\alpha} - \delta_{-\alpha\mu} x_{-\beta\nu}). \end{aligned}$$

2. *The natural representation of* $\mathfrak{sp}(n, \mathbb{C})$ *on* $W^{1,0}$ *is given by*

$$x_{\alpha\beta}\epsilon_\nu = \delta_{\beta\nu}\epsilon_\alpha + \operatorname{sign}(\nu\beta)\delta_{-\alpha\nu}\epsilon_{-\beta}.$$

Proof. By direct calculations. □

We shall discuss representations of $Sp(n)$ or $Sp(n, \mathbb{C})$. Let V be a finite dimensional irreducible unitary $Sp(n)$-module. We choose $\mathfrak{h} := \operatorname{span}_{\mathbb{R}}\{\sqrt{-1}x_{kk} | 1 \leq k \leq n\}$ as a maximal abelian subalgebra of $\mathfrak{sp}(n)$. Then we decompose V into simultaneous eigenspaces called weight spaces with respect to $\mathfrak{h}$ and have the highest weight $\rho = (\rho^1, \cdots, \rho^n)$ in weights for V. Here, ρ^k is the eigenvalue of x_{kk}. This highest weight ρ satisfies *the dominant integral condition*,

$$\rho = (\rho^1, \cdots, \rho^n) \in \mathbb{Z}^n, \quad \rho^1 \geq \rho^2 \geq \cdots \geq \rho^n \geq 0.$$

Conversely, for a dominant integral weight ρ, we have a unique irreducible unitary $Sp(n)$-module with highest weight ρ up to equivalence. Thus, all the irreducible unitary $Sp(n)$-modules are parametrized by dominant integral weights. So, we denote by (π_ρ, V_ρ) an irreducible unitary representation of $Sp(n)$ with highest weight ρ. For example, the natural representation of $Sp(n)$ on $W^{1,0}$ has highest weight $\mu_1 := (1, 0, \cdots, 0)$.

A significant feature common to irreducible $Sp(n)$-modules is that there is a real or quaternionic structure compatible with the action of $Sp(n)$. A *real* (resp. *quaternionic*) *structure* on complex vector space V is said to be an anti-linear automorphism $\mathfrak{J} : V \to V$ with $\mathfrak{J}^2 = 1$ (resp. $\mathfrak{J}^2 = -1$). We shall explain this feature refering to [5]. Let (π_ρ, V_ρ) be as above. Considering the contragradient representation π^*_ρ on dual space V^*_ρ, we know that the weights are $\{-\lambda \mid \lambda \text{ is weight of } (\pi_\rho, V_\rho)\}$ and the lowest one is $-\rho$. But, we can move this lowest weight $-\rho$ to ρ by the Weyl group of $Sp(n)$. Thus, the highest weight of (π^*_ρ, V^*_ρ) is also ρ and there is a $Sp(n)$-isomorphism $V_\rho \simeq V^*_\rho$. Now, we consider a vector space of the $Sp(n)$-invariant bilinear forms on V_ρ, that is, $\mathrm{Hom}_{Sp(n)}(V_\rho, V^*_\rho)$. It follows from Schur's lemma that $\mathrm{Hom}_{Sp(n)}(V_\rho, V^*_\rho)$ is a one-dimensional complex vector space. Then we take a non-zero $Sp(n)$-invariant bilinear form Ω on V_ρ, which is non-degenerate because of irreducibility of V_ρ. We define a symmetric form Ω_+ and a symplectic form Ω_- by $\Omega_\pm(\phi, \psi) := \Omega(\phi, \psi) \pm \Omega(\psi, \phi)$. As $\dim \mathrm{Hom}_{Sp(n)}(V_\rho, V^*_\rho)$ is one, we have either $\Omega_+ = 0$ or $\Omega_- = 0$. Thus, there is a unique non-degenerate symmetric or symplectic form on V_ρ up to a scalar multiple. On the other hands, there is a $Sp(n)$-isomorphism $\bar{V}_\rho \simeq V^*_\rho$ because of unitarity. Here, $(\bar{\pi}_\rho, \bar{V}_\rho)$ is the complex conjugate representation of (π_ρ, V_ρ). As a result, we have an isomorphism $V_\rho \simeq V^*_\rho \simeq \bar{V}_\rho$ as $Sp(n)$-modules. Normalizing the symmetric or symplectic form Ω, we have a unique $Sp(n)$-invariant real or quaternionic structure $\mathfrak{J} : V_\rho \to V_\rho$ such that $(\mathfrak{J}\phi, \mathfrak{J}\psi) = (\psi, \phi)$ for any ϕ and ψ in V_ρ.

Proposition 2.2. *On an irreducible unitary $Sp(n)$-module V_ρ, there is a unique $Sp(n)$-invariant real or quaternionic structure $\mathfrak{J}$ such that $(\mathfrak{J}\phi, \mathfrak{J}\psi) = (\psi, \phi)$ for any ϕ and ψ in V_ρ. Furthermore, if the highest weight ρ satisfies that $\sum_{k=1}^n \rho^k$ is even (resp. odd), then $\mathfrak{J}$ is real (resp. quaternionic).*

Proof. It remains to show that $\sum \rho^k$ is even (resp. odd) iff $\mathfrak{J}$ is real (resp. quaternionic). We consider the natural $Sp(n)$-module $W^{1,0}$ with $Sp(n)$-invariant quaternionic structure $\mathfrak{J}$. The tensor product module $\otimes^{2l} W^{1,0}$ has a real structure $\otimes^{2l}\mathfrak{J}$, and $\otimes^{2l+1} W^{1,0}$ has a quaternionic structure $\otimes^{2l+1}\mathfrak{J}$. Each irreducible $Sp(n)$-module V_ρ can be realized as a component of $\otimes^{2l} W^{1,0}$ or $\otimes^{2l+1} W^{1,0}$. In particular, V_ρ is in $\otimes^{2l} W^{1,0}$ if and only if $\sum \rho^k$ is even. Thus, we have proved the proposition. □

3. Casimir elements and Universal Bochner-Weitzenböck formulas

We introduce the enveloping algebra and Casimir elements of the Lie algebra $\mathfrak{sp}(n, \mathbb{C})$. Let $\{X_i\}_{i=1}^s$ be a basis of $\mathfrak{sp}(n, \mathbb{C})$. The enveloping algebra $U(\mathfrak{sp}(n, \mathbb{C}))$

is an associative algebra over $\mathbb{C}$ such that basis is $\{(X_1)^{k_1}\cdots(X_s)^{k_s} | k_i \geq 0\}$ and relations are $X_iX_j - X_jX_i = [X_i, X_j]$ for $i, j = 1, \cdots, s$. We denote by $\mathfrak{Z}$ the center of $U(\mathfrak{sp}(n,\mathbb{C}))$, which is characterized as the invariant sub-algebra of $U(\mathfrak{sp}(n,\mathbb{C}))$ under the adjoint action of $\mathfrak{sp}(n,\mathbb{C})$. We call elements in $\mathfrak{Z}$ *Casimir elements.*

We shall construct generators of $\mathfrak{Z}$. Let $\{x_{\alpha\beta} \mid \alpha, \beta = \pm1, \cdots, \pm n\}$ be the canonical elements of $\mathfrak{sp}(n,\mathbb{C})$ defined in (2.1). For each non-negative integer q, we define an element $x^q_{\alpha\beta}$ in $U(\mathfrak{sp}(n,\mathbb{C}))$ by

$$x^q_{\alpha\beta} := \begin{cases} \sum_{\alpha_1,\alpha_2,\cdots,\alpha_{q-1}=\pm1,\cdots,\pm n} x_{\alpha\alpha_1}x_{\alpha_1\alpha_2}\cdots x_{\alpha_{q-1}\beta} & q \geq 1, \\ \delta_{\alpha\beta} & q = 0. \end{cases}$$

It is easy to show that these elements satisfy

$$[x_{\mu\nu}, x^q_{\alpha\beta}] = \delta_{\alpha\nu}x^q_{\mu\beta} - \delta_{\beta\mu}x^q_{\alpha\nu} + \mathrm{sign}(\mu\nu)(\delta_{-\nu\beta}x^q_{\alpha-\mu} - \delta_{-\mu\alpha}x^q_{-\nu\beta}).$$

It follows that $c_q := \sum_\alpha x^q_{\alpha\alpha}$ is an invariant element under the adjoint action of $\mathfrak{sp}(n,\mathbb{C})$, namely, a Casimir element. The following fact for these Casimir elements has been known [13], [14], [15].

Proposition 3.1. *The Casimir elements $\{c_2, c_4, \cdots, c_{2n}\}$ generate the center $\mathfrak{Z}$ algebraically. In particular, the Casimir elements with odd degree are polynomials in $\{c_2, c_4, \cdots, c_{2n}\}$.*

A nice method of calculating eigenvalue of c_q on V_ρ has been shown in [14]. The idea is to use projection operators from $V_\rho \otimes W^{1,0}$ onto irreducible components. The projection operators give homomorphisms among $Sp(n)$-modules, which are just principal symbols of hyper-Kählerian gradients. We call these homomorphisms *Clifford homomorphisms.* Let us clarify the relationship between Clifford homomorphisms and Casimir elements, and calculate the eigenvalues of c_q.

Let V_ρ be an irreducible unitary $Sp(n)$-module with real or quaternionic structure $\mathfrak{J}$ such that $(\mathfrak{J}\phi, \mathfrak{J}\psi) = (\psi, \phi)$. We consider the natural representation $(\pi_{\mu_1}, W^{1,0})$ and the tensor product representation $(\pi_\rho \otimes \pi_{\mu_1}, V_\rho \otimes W^{1,0})$. It follows from a decomposition formula (see [14] or Chapter 7 in [5]) that highest weights of irreducible components in $V_\rho \otimes W^{1,0}$ are

$$\{\rho + \mu_\nu | \ \rho + \mu_\nu \text{ is dominant integral}, \ \nu = \pm1, \cdots, \pm n\},$$

where we set

$$\mu_k := (\underbrace{0, \cdots, 0}_{k-1}, 1, \underbrace{0, \cdots, 0}_{n-k}) \in \mathbb{Z}^n$$

and $\mu_{-k} := -\mu_k$ for $k = 1, \cdots, n$. Define $V_{\rho+\mu_\nu} := \{0\}$ for $\rho+\mu_\nu$ without dominant integral condition, and we can describe the irreducible decomposition as follows:

$$V_\rho \otimes W^{1,0} = \bigoplus_{\nu=\pm1,\cdots,\pm n} V_{\rho+\mu_\nu} = \bigoplus_{k=1,\cdots,n} V_{\rho+\mu_k} \oplus V_{\rho-\mu_k}.$$

We adopt a Hermitian inner product and a real or quaternionic structure on each component $V_{\rho+\mu_\nu}$ induced by the ones on $V_\rho \otimes W^{1,0}$. Note that each component has a real (resp. quaternionic) structure if V_ρ has a quaternionic (resp. real) one.

Definition 3.2. We define a linear mapping $p_\nu(u) : V_\rho \to V_{\rho+\mu_\nu}$ for u in $W^{1,0}$ by

$$W^{1,0} \times V_\rho \ni (u,\phi) \mapsto p_\nu(u)\phi := \Pi_\nu(\phi \otimes u) \in V_{\rho+\mu_\nu},$$

where Π_ν is the orthogonal projection from $V_\rho \otimes W^{1,0}$ onto $V_{\rho+\mu_\nu}$. We denote by $p_\nu(u)^*$ the adjoint of $p_\nu(u)$ with respect to inner products on V_ρ and $V_{\rho+\mu_\nu}$. By using real or quaternionic structures, we have other linear mappings

$$\mathfrak{J}p_\nu(u)\mathfrak{J}^{-1} : V_\rho \to V_{\rho+\mu_\nu}, \quad \mathfrak{J}p_\nu(u)^*\mathfrak{J}^{-1} : V_{\rho+\mu_\nu} \to V_\rho.$$

We call these linear mappings $p_\nu(u)$, $p_\nu(u)^*$, $\mathfrak{J}p_\nu(u)\mathfrak{J}^{-1}$, and $\mathfrak{J}p_\nu(u)^*\mathfrak{J}^{-1}$ *Clifford homomorphisms.*

The following lemma is useful.

Lemma 3.3. *The Clifford homomorphisms satisfy*

$$\begin{aligned} \mathfrak{J}p_\nu(u)\mathfrak{J}^{-1} = p_\nu(\mathfrak{J}u), \quad \mathfrak{J}p_\nu(u)^*\mathfrak{J}^{-1} = p_\nu(\mathfrak{J}u)^*, \\ (\mathfrak{J}p_\nu(u)\mathfrak{J}^{-1})^* = \mathfrak{J}p_\nu(u)^*\mathfrak{J}^{-1}, \quad \mathfrak{J}\mathfrak{J}p_\nu(u)\mathfrak{J}^{-1}\mathfrak{J}^{-1} = -p_\nu(u). \end{aligned} \tag{3.1}$$

Now, we can relate the Clifford homomorphisms to the enveloping algebra and Casimir elements of $\mathfrak{sp}(n,\mathbb{C})$.

Proposition 3.4. *We assign a constant w_ν to each component $V_{\rho+\mu_\nu}$,*

$$\begin{cases} w_{+k} := -(\rho^k - k + 1) & \text{for } V_{\rho+\mu_k}, \\ w_{-k} := \rho^k - k + 2n + 1 & \text{for } V_{\rho+\mu_{-k}} = V_{\rho-\mu_k}. \end{cases} \tag{3.2}$$

We call this constant w_ν the conformal weight associated to ρ and $\rho+\mu_\nu$. Then, we have the followings:

1.

$$\sum_{\alpha=\pm1,\cdots,\pm n} p_\nu(\epsilon_\alpha)^* p_\nu(\epsilon_\alpha) = \frac{\dim V_{\rho+\mu_\nu}}{\dim V_\rho}. \tag{3.3}$$

Here, $\{\epsilon_\alpha\}_\alpha$ is a unitary basis of $W^{1,0}$.

2. *For $q = 0, 1, \cdots,$*

$$\sum_{\nu=\pm1,\cdots,\pm n} w_\nu^q \, p_\nu(\epsilon_\alpha)^* p_\nu(\epsilon_\beta) = \mathrm{sign}(\alpha\beta)\pi_\rho(x^q_{-\alpha-\beta}), \tag{3.4}$$

$$\sum_{\nu=\pm1,\cdots,\pm n} w_\nu^q \, \mathfrak{J}p_\nu(\epsilon_\alpha)^* p_\nu(\epsilon_\beta)\mathfrak{J}^{-1} = \pi_\rho(x^q_{\alpha\beta}), \tag{3.5}$$

$$\sum_{\nu=\pm1,\cdots,\pm n} w_\nu^q \, \mathfrak{J}p_\nu(\epsilon_\alpha)^*\mathfrak{J}^{-1} p_\nu(\epsilon_\beta) = -\mathrm{sign}(\beta)\pi_\rho(x^q_{\alpha-\beta}), \tag{3.6}$$

$$\sum_{\nu=\pm1,\cdots,\pm n} w_\nu^q \, p_\nu(\epsilon_\alpha)^* \mathfrak{J}p_\nu(\epsilon_\beta)\mathfrak{J}^{-1} = -\mathrm{sign}(\alpha)\pi_\rho(x^q_{-\alpha\beta}). \tag{3.7}$$

3. *The eigenvalue of the Casimir element c_q on V_ρ is*

$$\pi_\rho(c_q) = \pi_\rho(\sum_\alpha x^q_{\alpha\alpha}) = \frac{1}{\dim V_\rho} \sum_{\nu=\pm1,\cdots,\pm n} w^q_\nu \dim V_{\rho+\mu_\nu}. \tag{3.8}$$

Proof. We can prove this proposition with a method given in [7], [8]. So, we give an outline of the proof. First, since Clifford homomorphisms are defined by the orthogonal projection $\Pi_\nu : V_\rho \otimes W^{1,0} \to V_{\rho+\mu_\nu}$, we have

$$\dim V_{\rho+\mu_\nu} = \mathrm{trace}(\Pi_\nu) = \dim V_\rho \times \sum_\alpha p_\nu(\epsilon_\alpha)^* p_\nu(\epsilon_\alpha).$$

Then we have proved (3.3). Next, since the inner product on $V_{\rho+\mu_\nu}$ is given along the tensor inner product on $V_\rho \otimes W^{1,0}$, we show that

$$\sum_\nu p_\nu(\epsilon_\alpha)^* p_\nu(\epsilon_\beta) = (\epsilon_\alpha, \epsilon_\beta) = \delta_{\alpha\beta}. \tag{3.9}$$

Let $c_2 := \sum x_{\alpha\beta} x_{\beta\alpha}$ be the second Casimir element. This Casimir element acts on V_ρ as $\pi_\rho(c_2) = 2(\|\delta+\rho\|^2 - \|\delta\|^2)\mathrm{id}$, where $\|\rho\|^2$ is $\sum_{k=1}^n (\rho^k)^2$ and $\delta = (n, n-1, \cdots, 1)$ is half sum of the positive roots of $\mathfrak{sp}(n, \mathbb{C})$. We define an operator C on $V_\rho \otimes W^{1,0}$ by

$$C := \{(\pi_\rho \otimes \pi_{\mu_1})(c_2) - \mathrm{id} \otimes \pi_{\mu_1}(c_2) - \pi_\rho(c_2) \otimes \mathrm{id}\}/4 = 1/2 \sum \pi_\rho(x_{\alpha\beta}) \otimes \pi_{\mu_1}(x_{\beta\alpha}).$$

This operator acts on irreducible component $V_{\rho+\mu_\nu} \subset V_\rho \otimes W^{1,0}$ as constant multiple $-w_\nu\mathrm{id}$, where

$$-w_\nu = (\|\delta + \rho + \mu_\nu\|^2 - \|\delta + \rho\|^2 - 2n - 1)/2.$$

Then we have

$$\begin{aligned}
\sum_\nu -w_\nu p_\nu(\epsilon_\alpha)\phi = C(\phi \otimes \epsilon_\alpha) &= 1/2 \sum \pi_\rho(x_{\beta\nu})\phi \otimes \pi_{\mu_1}(x_{\nu\beta})\epsilon_\alpha \\
&= -\mathrm{sign}(\alpha\beta) \sum_\beta \pi_\rho(x_{-\beta-\alpha})\phi \otimes \epsilon_\beta \\
&= \sum_\nu (-\sum_\beta \mathrm{sign}(\alpha\beta) p_\nu(\epsilon_\beta) \pi_\rho(x_{-\beta-\alpha})\phi),
\end{aligned}$$

for $\phi \otimes \epsilon_\alpha$ in $V_\rho \otimes W^{1,0}$. Thus we get

$$w_\nu p_\nu(\epsilon_\alpha) = \sum_\beta \mathrm{sign}(\alpha\beta) p_\nu(\epsilon_\beta) \pi_\rho(x_{-\beta-\alpha}). \tag{3.10}$$

We use this relation q-times and (3.9). Then we have (3.4). For other cases (3.5)-(3.7), we use $\mathfrak{J}(\epsilon_\alpha) = \mathrm{sign}(\alpha)\epsilon_{-\alpha}$ and (3.1). Taking the trace in (3.4), we have the eigenvalue of c_q on V_ρ as (3.8). □

We consider the equation (3.4) for $q = 0, 1, \cdots, 2n-1$. Since the conformal weights $\{w_\nu\}_{\nu=\pm1}^{\pm n}$ are different from each other, the Clifford homomorphism $p_\nu(\epsilon_\alpha)^* p_\nu(\epsilon_\beta)$ is realized as a linear combination of $\{\pi_\rho(x^p_{-\alpha-\beta})\}_{p=0}^{2n-1}$. Therefore, if there exist relations for $\{x^q_{\alpha\beta}\}_{q\geq 0}$, then we have relations among Clifford homomorphisms. In fact, Proposition 3.1 suggests the existence of such relations. In the rest of this section, we search explicit relations among $\{x^q_{\alpha\beta}\}_{q\geq 0}$ and $\{c_q\}_{q\geq 0}$. The point is that we try to exchange indices α and β of $x^q_{\alpha\beta}$. To make calculations easy, we define translated elements $\{\hat{x}^q_{\alpha\beta}\}_{q\geq 0}$ in $U(\mathfrak{sp}(n,\mathbb{C}))$ by

$$\hat{x}_{\alpha\beta} := x_{\alpha\beta} - (n+1/2)\delta_{\alpha\beta},$$

and

$$\hat{x}^q_{\alpha\beta} := \begin{cases} \sum_{\alpha_1,\alpha_2,\cdots,\alpha_{q-1}=\pm 1,\cdots,\pm n} \hat{x}_{\alpha\alpha_1}\hat{x}_{\alpha_1\alpha_2}\cdots\hat{x}_{\alpha_{q-1}\beta} & q\geq 1, \\ \delta_{\alpha\beta} & q=0, \end{cases}$$

where we remark that

$$\hat{x}^q_{\alpha\beta} = \sum_{p=0}^{q}\binom{q}{p}(-n-1/2)^{q-p}x^p_{\alpha\beta}.$$

The translated elements satisfy

$$\hat{x}_{\alpha\beta} = -\mathrm{sign}(\alpha\beta)\hat{x}_{-\beta-\alpha} - (2n+1)\delta_{\alpha\beta}, \tag{3.11}$$

$$[\hat{x}_{\mu\nu},\hat{x}^q_{\alpha\beta}] = \delta_{\alpha\nu}\hat{x}^q_{\mu\beta} - \delta_{\beta\mu}\hat{x}^q_{\alpha\nu} + \mathrm{sign}(\mu\nu)(\delta_{-\nu\beta}\hat{x}^q_{\alpha-\mu} - \delta_{-\mu\alpha}\hat{x}^q_{-\nu\beta}), \tag{3.12}$$

$$\sum_\nu \hat{x}^p_{\alpha\nu}\hat{x}^q_{\nu\beta} = \hat{x}^{p+q}_{\alpha\beta}. \tag{3.13}$$

We state the main theorem.

Theorem 3.5 (universal Bochner-Weitzenböck formulas)**.** *The translated element $\hat{x}^q_{\alpha\beta}$ is represented as a linear combination of $\{\hat{x}^p_{-\beta-\alpha}\}_{p=0}^q$ whose coefficients are Casimir elements,*

(3.14)

$$\hat{x}^q_{\alpha\beta} = \mathrm{sign}(\alpha\beta)\{(-1)^q\hat{x}^q_{-\beta-\alpha} - \frac{1-(-1)^q}{2}\hat{x}^{q-1}_{-\beta-\alpha} - \sum_{p=0}^{q-1}(-1)^p\hat{c}_{q-1-p}\hat{x}^p_{-\beta-\alpha}\}.$$

Taking the trace in the above equation, we have identities among Casimir elements $\{\hat{c}_p := \sum_\alpha \hat{x}^p_{\alpha\alpha}\}_{p\geq 0}$,

$$2\hat{c}_{2q+1} = -\hat{c}_{2q} - \sum_{p=0}^{2q}(-1)^p\hat{c}_{2q-p}\hat{c}_p, \tag{3.15}$$

for $q = 0,1,\cdots$.

Proof. It follows from (3.12) and (3.13) that

$$\hat{x}^{q+1}_{\alpha\beta} - \sum_\nu \hat{x}^q_{\nu\beta}\hat{x}_{\alpha\nu} = \sum_\nu[\hat{x}_{\alpha\nu},\hat{x}_{\nu\beta}] = (2n+1)\hat{x}^q_{\alpha\beta} - \delta_{\alpha\beta}\hat{c}_q - \mathrm{sign}(\alpha\beta)\hat{x}^q_{-\beta-\alpha}.$$

Substituting (3.11) into this equation, we have

$$\hat{x}^{q+1}_{\alpha\beta} = -\delta_{\alpha\beta}\hat{c}_q - \mathrm{sign}(\alpha\beta)\hat{x}^q_{-\beta-\alpha} - \sum_\nu \mathrm{sign}(\nu\alpha)\hat{x}^q_{\nu\beta}\hat{x}_{-\nu-\alpha}. \tag{3.16}$$

This equation means that $\hat{x}^q_{\alpha\beta}$ is a linear combination of $\{\hat{x}^p_{-\beta-\alpha}\}_{p=0}^q$ whose coefficients are Casimir elements given inductively. Then we set

$$\hat{x}^q_{\alpha\beta} := -\mathrm{sign}(\alpha\beta)\sum_{p=0}^{q} a_{q,p}(\hat{c})\hat{x}^p_{-\beta-\alpha}$$

for $q \geq 0$, where $\{a_{q,p}(\hat{c})\}_{q\geq p\geq 0}$ are polynomials in Casimir elements $\{\hat{c}_0, \hat{c}_1, \cdots\}$. Substituting this into (3.16), we obtain a recursion formula of $\{a_{q,p}(\hat{c})\}_{q\geq p\geq 0}$,

$$a_{q+1,p}(\hat{c}) = \begin{cases} -a_{q,q}(\hat{c}) & \text{for } p = q+1, \\ -a_{q,q-1}(\hat{c}) + 1 & \text{for } p = q, \\ -a_{q,p-1}(\hat{c}) & \text{for } 1 \leq p \leq q-1, \\ \hat{c}_q & \text{for } p = 0. \end{cases}$$

Here, $a_{0,0}(\hat{c}) = -1$, $a_{1,0}(\hat{c}) = 2n+1 = \hat{c}_0 + 1$, $a_{1,1}(\hat{c}) = 1$. We solve this recursion formula and have (3.14). If we take the trace of (3.14) for the case that q is even, then we have trivial identities. When q is odd, we have (3.15). $\square$

Since this theorem induces Bochner-Weitzenböck formulas for hyper-Kählerian gradients in Section 5, we call the equations (3.14) *the universal Bochner-Weitzenböck formulas.*

We shall apply the universal Bochner-Weitzenböck formulas to constructing relations among Clifford homomorphisms. The next lemma follows from Proposition 3.4.

Lemma 3.6. *We define the translated conformal weight $\hat{w}_\nu$ by $\hat{w}_\nu := w_\nu - (n+1/2)$. Then we have*

$$(3.17)\quad \begin{aligned} \sum_{\nu=\pm1,\cdots,\pm n} \hat{w}^q_\nu\, p_\nu(\epsilon_\alpha)^* p_\nu(\epsilon_\beta) &= \mathrm{sign}(\alpha\beta)\pi_\rho(\hat{x}^q_{-\alpha-\beta}), \\ \sum_{\nu=\pm1,\cdots,\pm n} \hat{w}^q_\nu\, \mathfrak{J} p_\nu(\epsilon_\alpha)^* p_\nu(\epsilon_\beta)\mathfrak{J}^{-1} &= \pi_\rho(\hat{x}^q_{\alpha\beta}), \\ \sum_{\nu=\pm1,\cdots,\pm n} \hat{w}^q_\nu\, \mathfrak{J} p_\nu(\epsilon_\alpha)^* \mathfrak{J}^{-1} p_\nu(\epsilon_\beta) &= -\mathrm{sign}(\beta)\pi_\rho(\hat{x}^q_{\alpha-\beta}), \\ \sum_{\nu=\pm1,\cdots,\pm n} \hat{w}^q_\nu\, p_\nu(\epsilon_\alpha)^* \mathfrak{J} p_\nu(\epsilon_\beta)\mathfrak{J}^{-1} &= -\mathrm{sign}(\alpha)\pi_\rho(\hat{x}^q_{-\alpha\beta}), \end{aligned}$$

and

$$\pi_\rho(\hat{c}_q) = \pi_\rho(\sum_\alpha \hat{x}^q_{\alpha\alpha}) = \frac{1}{\dim V_\rho}\sum_{\nu=\pm1,\cdots,\pm n} \hat{w}^q_\nu\, \dim V_{\rho+\mu_\nu}.$$

From the universal Bochner-Weitzenböck formulas (3.14) and the equations (3.17), we have identities for Clifford homomorphisms,

$$\sum_\nu \left\{(1-(-1)^q)\hat{w}_\nu^q + \frac{1-(-1)^q}{2}\hat{w}_\nu^{q-1} + \sum_{p=0}^{q-1}(-1)^p \pi_\rho(\hat{c}_{q-1-p})\hat{w}_\nu^p\right\}(p_\nu(u)^* p_\nu(u) + \mathfrak{J} p_\nu(u)^* p_\nu(u)\mathfrak{J}^{-1}) = 0 \tag{3.18}$$

for $q = 1, 2, \cdots$ and any u in $W^{1,0}$. We shall detect independent identities in (3.18). For $s \gg 0$, we consider (3.18) for $q = 1, \cdots, 2s$. We represent the coefficients of $\{p_\nu(u)^* p_\nu(u) + \mathfrak{J} p_\nu(u)^* p_\nu(u)\mathfrak{J}^{-1}\}_\nu$ by a $2s \times 2n$ matrix $C(s) \cdot W(s)$, where $C(s)$ is a $2s \times 2s$ matrix,

$$C(s) := \begin{pmatrix} \pi_\rho(\hat{c}_0)+1 & 2 & 0 & 0 & \cdots & \cdots & 0 \\ \pi_\rho(\hat{c}_1) & -\pi_\rho(\hat{c}_0) & 0 & 0 & \cdots & \cdots & 0 \\ \pi_\rho(\hat{c}_2) & -\pi_\rho(\hat{c}_1) & \pi_\rho(\hat{c}_0)+1 & 2 & 0 & \cdots & 0 \\ \pi_\rho(\hat{c}_3) & -\pi_\rho(\hat{c}_2) & \pi_\rho(\hat{c}_1) & -\pi_\rho(\hat{c}_0) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -\pi_\rho(\hat{c}_0) \end{pmatrix},$$

and $W(s)$ is a $2s \times 2n$ matrix,

$$W(s) := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \hat{w}_{-n} & \hat{w}_{-n+1} & \cdots & \hat{w}_n \\ (\hat{w}_{-n})^2 & (\hat{w}_{-n+1})^2 & \cdots & (\hat{w}_n)^2 \\ \cdots & \cdots & \cdots & \cdots \\ (\hat{w}_{-n})^{2s-1} & (\hat{w}_{-n+1})^{2s-1} & \cdots & (\hat{w}_n)^{2s-1} \end{pmatrix}.$$

Because $\pi_\rho(\hat{c}_0) = 2n$ and $\pi_\rho(\hat{c}_1) = -n(2n+1)$, we should consider only the even cases in (3.18), that is,

$$\sum_\nu \{\sum_{p=0}^{2q-1}(-1)^p \pi_\rho(\hat{c}_{2q-1-p})\hat{w}_\nu^p\}(p_\nu(u)^* p_\nu(u) + \mathfrak{J} p_\nu(u)^* p_\nu(u)\mathfrak{J}^{-1}) = 0 \tag{3.19}$$

for $q = 1, \cdots, s$. There is a possibility that these identities are still linear dependent. To simplify a discussion, we assume that the number of irreducible components in $V_\rho \otimes W^{1,0}$ is $2n$. In (3.19), the coefficients of $\{p_\nu(u)^* p_\nu(u) + \mathfrak{J} p_\nu(u)^* p_\nu(u)\mathfrak{J}^{-1}\}_\nu$ are represented by a $s \times 2n$ matrix $D(s) \cdot W(s)$, where $D(s)$ is a $s \times 2s$ matrix,

$$D(s) := \begin{pmatrix} \pi_\rho(\hat{c}_1) & -\pi_\rho(\hat{c}_0) & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \pi_\rho(\hat{c}_3) & -\pi_\rho(\hat{c}_2) & \pi_\rho(\hat{c}_1) & -\pi_\rho(\hat{c}_0) & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \pi_\rho(\hat{c}_1) & -\pi_\rho(\hat{c}_0) \end{pmatrix}.$$

Since the rank of $W(s)$ is $2n$ and the rank of $D(s)$ is s, we have

$$2n - s \le \mathrm{rank}(D(s) \cdot W(s)) \le \min\{s, 2n\}.$$

In particular, the rank of $D(n)\cdot W(n)$ is just n. Thus there are at least n independent identities in (3.19). In general, if the number of irreducible components in $V_\rho \otimes W^{1,0}$ is N,

$$N = \#\{\rho+\mu_\nu |\ \rho \text{ is dominant integral}, \nu=\pm 1,\cdots,\pm n\} \le 2n,$$

then we show that there are at least $[N/2]$ independent identities in (3.19).

On the other hand, from (3.14) and (3.17), we also have

$$\sum_\nu \Big\{(1+(-1)^q)\hat{w}_\nu^q - \frac{1-(-1)^q}{2}\hat{w}_\nu^{q-1} - \sum_{p=0}^{q-1}(-1)^p\pi_\rho(\hat{c}_{q-1-p})\hat{w}_\nu^p\Big\}(p_\nu(u)^*p_\nu(u) - \mathfrak{J}p_\nu(u)^*p_\nu(u)\mathfrak{J}^{-1}) = 0,$$

for $q=0,1,\cdots$. In a similar discussion, we know that there are at least $[\frac{N+1}{2}]$ independent identities in the above. We can also have some independent identities for $\{\mathfrak{J}p_\nu(u)^*\mathfrak{J}^{-1}p_\nu(u)\}_\nu$ and $\{p_\nu(u)^*\mathfrak{J}p_\nu(u)\mathfrak{J}^{-1}\}_\nu$.

Proposition 3.7. *Let V_ρ be an irreducible unitary $Sp(n)$-module such that the number of irreducible components in $V_\rho\otimes W^{1,0}$ is N. We consider the Clifford homomorphisms $\{p_\nu, p_\nu^*, \mathfrak{J}p_\nu\mathfrak{J}^{-1}, \mathfrak{J}p_\nu^*\mathfrak{J}^{-1}\}_{\nu=\pm1}^{\pm n}$, where we set $p_\nu=0$ unless $\rho+\mu_\nu$ is dominant integral. Then we have the following independent identities among the Clifford homomorphisms.*

1. $$\sum_\nu p_\nu(u)^*p_\nu(u) = (u,u).$$

2. *For $q=1,\cdots,[N/2]$,*

$$\sum_\nu \{\sum_{p=0}^{2q-1}(-1)^p\pi_\rho(\hat{c}_{2q-1-p})\hat{w}_\nu^p\}(p_\nu(u)^*p_\nu(u) + \mathfrak{J}p_\nu(u)^*p_\nu(u)\mathfrak{J}^{-1}) = 0.$$

3. *For $q=0,1,\cdots,[\frac{N+1}{2}]-1$,*

$$\sum_\nu \{\hat{w}_\nu^{2q} - \sum_{p=0}^{2q-1}(-1)^p\pi_\rho(\hat{c}_{2q-1-p})\hat{w}_\nu^p\}(p_\nu(u)^*p_\nu(u) - \mathfrak{J}p_\nu(u)^*p_\nu(u)\mathfrak{J}^{-1}) = 0,$$

$$\sum_\nu \{\hat{w}_\nu^{2q} - \sum_{p=0}^{2q-1}(-1)^p\pi_\rho(\hat{c}_{2q-1-p})\hat{w}_\nu^p\}\mathfrak{J}p_\nu(u)^*\mathfrak{J}^{-1}p_\nu(u) = 0,$$

$$\sum_\nu \{\hat{w}_\nu^{2q} - \sum_{p=0}^{2q-1}(-1)^p\pi_\rho(\hat{c}_{2q-1-p})\hat{w}_\nu^p\}p_\nu(u)^*\mathfrak{J}p_\nu(u)\mathfrak{J}^{-1} = 0.$$

4. Hyper-Hermitian gradients and their conformal covariance

A real $4n$-dimensional Riemannian manifold (M, g) is said to be an *almost hyper-Hermitian manifold* if M has three almost complex structures I, J and K such that I and J are orthogonal with respect to the metric g, and $IJ = -JI = K$. If I, J and K are parallel with respect to the Levi-Civita connection, we call (M, g, I, J, K) a *hyper-Kähler manifold*, [9].

We consider an almost hyper-Hermitian manifold (M, g, I, J, K). Some results about hyper-Hermitian structure on vector spaces in Section 2 can be extended globally on M. We decompose the tangent bundle $T(M) \otimes \mathbb{C}$ and the cotangent bundle $T^*(M) \otimes \mathbb{C}$ with respect to I,

$$T(M) \otimes \mathbb{C} = T^{1,0}(M) \oplus T^{0,1}(M), \quad T^*(M) \otimes \mathbb{C} = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M).$$

These vector bundles have Hermitian metrics $(u, v) := g(u, \bar{v})$. Another almost complex structure J induces bundle isomorphisms $J : T^{1,0}(M) \to T^{0,1}(M)$ and $J : T^{0,1}(M) \to T^{1,0}(M)$. Then we have a quaternionic structure $\mathfrak{J}$ on each bundle defined by $\mathfrak{J}(u) := J(\bar{u})$. These structure produce bundle isomorphisms,

$$\Lambda^{1,0}(M) \overset{(\cdot,\cdot)}{\simeq} T^{0,1}(M) \overset{J}{\simeq} T^{1,0}(M) \overset{(\cdot,\cdot)}{\simeq} \Lambda^{0,1}(M). \tag{4.1}$$

The structure group of the frame bundle on M reduces to the symplectic group $Sp(n)$. We denote the principal $Sp(n)$-bundle on M by $\mathbf{Sp}(M)$. We consider the Levi-Civita connection ∇^{so} on $T(M)$ and define a connection ∇^{sp} by

$$\nabla^{sp}_X Y := \{\nabla^{so}_X Y - I(\nabla^{so}_X(IY)) - J(\nabla^{so}_X(JY)) - K(\nabla^{so}_X(KY))\}/4.$$

Since $\nabla^{sp} g = 0$ and $\nabla^{sp} I = \nabla^{sp} J = \nabla^{sp} K = 0$, the connection ∇^{sp} is a connection on $\mathbf{Sp}(M)$. Notice that the torsion tensor of ∇^{sp} is zero iff M is a hyper-Kähler manifold.

For an irreducible unitary $Sp(n)$-module V_ρ, we have a Hermitian vector bundle $\mathbf{S}_\rho := \mathbf{Sp}(M) \times_\rho V_\rho$ associated to $\mathbf{Sp}(M)$. We have known that there is a unique $Sp(n)$-invariant real or quaternionic structure $\mathfrak{J}$ on V_ρ. Because of the $Sp(n)$-invariance, we have a real or quaternionic structure $\mathfrak{J}$ on $\mathbf{S}_\rho$ such that $(\mathfrak{J}_x\phi, \mathfrak{J}_x\psi) = (\psi, \phi)$ for ϕ and ψ in $(\mathbf{S}_\rho)_x$. We consider a covariant derivative ∇ on $\mathbf{S}_\rho$ induced from the canonical connection ∇^{sp}. This covariant derivative ∇ splits into the sum of $\nabla^{1,0}$ and $\nabla^{0,1}$,

$$\begin{aligned}
\nabla^{1,0} &: \Gamma(\mathbf{S}_\rho) \xrightarrow{\nabla^{1,0}} \Gamma(\mathbf{S}_\rho \otimes \Lambda^{1,0}(M)) = \Gamma(\mathbf{S}_\rho \otimes T^{1,0}(M)), \\
\nabla^{0,1} &: \Gamma(\mathbf{S}_\rho) \xrightarrow{\nabla^{0,1}} \Gamma(\mathbf{S}_\rho \otimes \Lambda^{0,1}(M)) = \Gamma(\mathbf{S}_\rho \otimes T^{1,0}(M)),
\end{aligned}$$

where we use isomorphisms (4.1). Because $T^{1,0}(M)$ and $\mathbf{S}_\rho$ are associated vector bundles, we have the irreducible decomposition

$$\mathbf{S}_\rho \otimes T^{1,0}(M) = \bigoplus_{\nu=\pm 1,\cdots,\pm n} \mathbf{S}_{\rho+\mu_\nu} = \bigoplus_{k=1,\cdots,n} \mathbf{S}_{\rho+\mu_k} \oplus \mathbf{S}_{\rho-\mu_k}.$$

We divide $\nabla^{1,0}$ along this decomposition and have first-order differential operators

$$D_\nu : \Gamma(\mathbf{S}_\rho) \xrightarrow{\nabla^{1,0}} \Gamma(\mathbf{S}_\rho \otimes \Lambda^{1,0}(M)) = \Gamma(\mathbf{S}_\rho \otimes T^{1,0}(M)) \xrightarrow{\Pi_\nu} \Gamma(\mathbf{S}_{\rho+\mu_\nu}).$$

for $\nu = \pm 1, \cdots, \pm n$. We shall express the operator D_ν by using local frame of M. Let $\{\epsilon_\alpha\}_\alpha$ and $\{\bar{\epsilon}_\alpha\}_\alpha$ be local unitary frames of $T^{1,0}(M)$ and $T^{0,1}(M)$ given as (2.1). Then,

$$\begin{aligned} D_\nu(\phi) &= \Pi_\nu(\nabla^{1,0}\phi) = \Pi_\nu(\sum_\alpha \nabla_{\epsilon_\alpha}\phi \otimes \epsilon_\alpha^*) \\ &= \Pi_\nu(\sum_\alpha \nabla_{\epsilon_\alpha}\phi \otimes -\mathrm{sign}(\alpha)\epsilon_{-\alpha}) = -\sum_\alpha \mathrm{sign}(\alpha) p_\nu(\epsilon_{-\alpha})\nabla_{\epsilon_\alpha}\phi. \end{aligned}$$

We twist D_ν by $\mathfrak{J}$ and have another first order differential operator $\mathfrak{J}D_\nu\mathfrak{J}^{-1}$ whose local expression is

$$\mathfrak{J}D_\nu\mathfrak{J}^{-1} = \sum_\alpha p_\nu(\epsilon_\alpha)\nabla_{\bar{\epsilon}_\alpha}.$$

The twisted operator is just the operator $\Pi_\nu \circ \nabla^{0,1}$ as a component of $\nabla^{0,1}$,

$$\mathfrak{J}D_\nu\mathfrak{J}^{-1} : \Gamma(\mathbf{S}_\rho) \xrightarrow{\nabla^{0,1}} \Gamma(\mathbf{S}_\rho \otimes \Lambda^{0,1}(M)) = \Gamma(\mathbf{S}_\rho \otimes T^{1,0}(M)) \xrightarrow{\Pi_\nu} \Gamma(\mathbf{S}_{\rho+\mu_\nu}).$$

We call these first-order differential operators $\{D_\nu, \mathfrak{J}D_\nu\mathfrak{J}^{-1}\}_\nu$ *hyper-Hermitian gradients*. When M is a hyper-Kähler manifold, we call them *hyper-Kählerian gradients*.

In the rest of this section, we see conformal covariance of hyper-Hermitian gradients. Let D_ν be a hyper-Hermitian gradient with respect to the canonical connection ∇^{sp} on an almost hyper-Hermitian manifold (M, g, I, J, K). When we deform the metric g conformally to $g' = e^{2\sigma}g$ for σ in $C^\infty(M)$, the Riemannian manifold (M, g', I, J, K) is also an almost hyper-Hermitian manifold. Then we have another hyper-Hermitian gradient $D'_\nu : \Gamma(\mathbf{S}'_\rho) \to \Gamma(\mathbf{S}'_{\rho+\mu_\nu})$. The conformal deformation gives a principal bundle isomorphism

$$\mathbf{Sp}(M) \ni p = \{\epsilon_\alpha\}_\alpha \mapsto p' = \{\epsilon'_\alpha\}_\alpha = \{e^{-\sigma}\epsilon_\alpha\}_\alpha \in \mathbf{Sp}(M)'.$$

Here, we remark that $(\epsilon'_\alpha)^* = e^\sigma \epsilon_\alpha^*$. This principal bundle isomorphism induces a bundle isomorphism,

$$\psi_\rho : \mathbf{S}_\rho = \mathbf{Sp}(M) \times_\rho V_\rho \ni [p, \phi] \mapsto [p', \phi] \in \mathbf{Sp}(M)' \times_\rho V_\rho = \mathbf{S}'_\rho.$$

Since, precisely speaking, the Clifford homomorphism is defined through not the projection $\Pi_\nu : \mathbf{S}_\rho \otimes T^{1,0}(M) \to \mathbf{S}_{\rho+\mu_\nu}$ but $\Pi_\nu : \mathbf{S}_\rho \otimes \Lambda^{1,0}(M) \to \mathbf{S}_{\rho+\mu_\nu}$, we have

$$p_\nu(\epsilon_\alpha) = e^{-\sigma}\psi_{\rho+\mu_\nu} p_\nu(\epsilon_\alpha)\psi_\rho^{-1} : \mathbf{S}'_\rho \to \mathbf{S}'_{\rho+\mu_\nu}.$$

The following proposition answers the reason why we call w_ν the conformal weight.

Proposition 4.1. *Let D_ν be a hyper-Hermitian gradient with respect to the canonical connection ∇^{sp} on an almost hyper-Hermitian manifold (M, g, I, J, K). Deform the Riemannian metric g conformally to $g' = e^{2\sigma}g$, and consider a hyper-Hermitian gradient D'_ν on (M, g', I, J, K). Then we have*

$$D'_\nu = e^{(-w_\nu/2-1)\sigma}\psi_{\rho+\mu_\nu} \circ D_\nu \circ (e^{(-w_\nu/2)\sigma}\psi_\rho)^{-1}.$$

Here, w_ν is the conformal weight associated to ρ and $\rho+\mu_\nu$.

Proof. The proof is similar to the one for the conformal covariance of the Dirac operator [12]. First we can show that

$$[D_\nu, f] = -p_\nu\big(\sum \mathrm{sign}(\alpha)(\epsilon_\alpha f)\epsilon_{-\alpha}\big) \tag{4.2}$$

for f in $C^\infty(M)$. Next, we consider a local connection 1-form ω_α^β on $T^{1,0}(M)$ given by $\nabla^{sp}_V\epsilon_\alpha = \sum \omega_\alpha^\beta(V)\epsilon_\beta$ for each vector field V. It follows from the definition of ∇^{sp} that

$$\begin{aligned}\omega_\alpha^\beta(V) &= g(\nabla^{sp}_V\epsilon_\alpha, \bar\epsilon_\beta) = 1/2\{g(\nabla^{so}_V\epsilon_\alpha, \bar\epsilon_\beta) - \mathrm{sign}(\alpha\beta)g(\nabla^{so}_V\epsilon_{-\beta}, \bar\epsilon_{-\alpha})\}\\ &= -\mathrm{sign}(\alpha\beta)\omega_{-\beta}^{-\alpha}(V).\end{aligned}$$

Thus, ω_α^β is $\mathfrak{sp}(n,\mathbb{C})$-valued. Then the covariant derivative ∇ on $\mathbf{S}_\rho$ is

$$\nabla_V = 1/2\sum \omega_\alpha^\beta(V)\pi_\rho(\epsilon_\beta\otimes\epsilon_\alpha^* - \mathrm{sign}(\alpha\beta)\epsilon_{-\alpha}\otimes\epsilon_{-\beta}^*) = 1/2\sum\omega_\alpha^\beta(V)\pi_\rho(x_{\beta\alpha}).$$

On the other hand, the deformed Levi-Civita connection ∇'^{so} is

$$\nabla'^{so}_V W = \nabla^{so}_V W + (V\sigma)W + (W\sigma)V - g(V,W)\mathrm{grad}(\sigma)$$

for vector fields V and W. As a result, we have

$$\nabla'_V = \psi_\rho\circ\{\nabla_V + 1/2\sum_{\alpha\beta}((\epsilon_\alpha\sigma)g(V,\bar\epsilon_\beta)\pi_\rho(x_{\beta\alpha}) - (\bar\epsilon_\beta\sigma)g(V,\epsilon_\alpha)\pi_\rho(x_{\beta\alpha}))\}\circ\psi_\rho^{-1}.$$

Then,

$$\begin{aligned}D'_\nu &= -\sum\mathrm{sign}(\alpha)p_\nu(\epsilon'_{-\alpha})\nabla'_{\epsilon'_\alpha} = -\sum\mathrm{sign}(\alpha)p_\nu(\epsilon_{-\alpha})\nabla'_{\epsilon_\alpha}\\ &= e^{-\sigma}\psi_{\rho+\mu_\nu}\circ\{-\sum_\alpha\mathrm{sign}(\alpha)p_\nu(\epsilon_{-\alpha})(\nabla_{\epsilon_\alpha} + 1/2\sum_\beta(\epsilon_\beta\sigma)\pi_\rho(x_{\alpha\beta}))\}\circ\psi_\rho^{-1}\\ &= e^{-\sigma}\psi_{\rho+\mu_\nu}\circ\{D_\nu - 1/2\sum_{\alpha\beta}\mathrm{sign}(\alpha)(\epsilon_\beta\sigma)p_\nu(\epsilon_{-\alpha})\pi_\rho(x_{\alpha\beta})\}\circ\psi_\rho^{-1}\\ &= e^{-\sigma}\psi_{\rho+\mu_\nu}\circ\{D_\nu - w_\nu/2\,p_\nu(\sum_\beta\mathrm{sign}(\beta)(\epsilon_\beta\sigma)\epsilon_{-\beta})\}\circ\psi_\rho^{-1}\quad(\text{by (3.10)})\\ &= e^{(-w_\mu/2-1)\sigma}\psi_{\rho+\mu_\nu}\circ D_\nu\circ(e^{(-w_\nu/2)\sigma}\psi_\rho)^{-1}\quad(\text{by (4.2)}).\end{aligned}$$

Thus we have proved the proposition. □

5. Bochner-Weitzenböck formulas for hyper-Kählerian gradients

In this section, we assume that (M,g,I,J,K) is a hyper-Kähler manifold, namely, $\nabla^{sp} = \nabla^{so}$. On a hyper-Kähler manifold, the formal adjoint operators of D_ν and

$\mathfrak{J}D_\nu\mathfrak{J}^{-1}$ have local expressions as follows:

$$D_\nu^* = \sum_\alpha \mathrm{sign}(\alpha) p_\nu(\epsilon_{-\alpha})^* \nabla_{\bar\epsilon_\alpha} : \Gamma(\mathbf{S}_{\rho+\mu_\nu}) \to \Gamma(\mathbf{S}_\rho),$$
$$(\mathfrak{J}D_\nu\mathfrak{J}^{-1})^* = \mathfrak{J}D_\nu^*\mathfrak{J}^{-1} = -\sum_\alpha p_\nu(\epsilon_\alpha)^* \nabla_{\epsilon_\alpha} : \Gamma(\mathbf{S}_{\rho+\mu_\nu}) \to \Gamma(\mathbf{S}_\rho).$$

Thus, on a hyper-Kähler manifold, the principal symbols of hyper-Kählerian gradients and their adjoints are realized with Clifford homomorphisms. Then we can use results in Section 3.

Proposition 5.1. *We define a second-order derivative $\nabla^2_{V,W}$ by $\nabla_V\nabla_W - \nabla_{\nabla_V W}$ for vector fields V and W. Then hyper-Kählerian gradients satisfy that*

$$\begin{aligned} \sum_\nu \hat w_\nu^q D_\nu^* D_\nu &= -\sum_{\alpha,\beta} \pi_\rho(\hat x^q_{\alpha\beta}) \nabla^2_{\bar\epsilon_\alpha,\epsilon_\beta}, \\ \sum_\nu \hat w_\nu^q \mathfrak{J}D_\nu^* D_\nu \mathfrak{J}^{-1} &= -\sum_{\alpha,\beta} \mathrm{sign}(\alpha\beta)\pi_\rho(\hat x^q_{-\alpha-\beta}) \nabla^2_{\epsilon_\alpha,\bar\epsilon_\beta}, \\ \sum_\nu \hat w_\nu^q \mathfrak{J}D_\nu^* \mathfrak{J}^{-1} D_\nu &= -\sum_{\alpha,\beta} \mathrm{sign}(\alpha)\pi_\rho(\hat x^q_{-\alpha\beta}) \nabla^2_{\epsilon_\alpha,\epsilon_\beta}, \\ \sum_\nu \hat w_\nu^q D_\nu^* \mathfrak{J} D_\nu \mathfrak{J}^{-1} &= -\sum_{\alpha,\beta} \mathrm{sign}(\beta)\pi_\rho(\hat x^q_{\alpha-\beta}) \nabla^2_{\bar\epsilon_\alpha,\bar\epsilon_\beta}. \end{aligned} \tag{5.1}$$

Here, $\{\epsilon_\alpha\}_\alpha$ is local unitary frame and

$$\hat x_{\alpha\beta} := \epsilon_\alpha \otimes \epsilon_\beta^* - \mathrm{sign}(\alpha\beta)\epsilon_{-\beta}\otimes\epsilon^*_{-\alpha} - (n+1/2)\delta_{\alpha\beta}$$

is a local section of the enveloping algebra bundle $\mathbf{Sp}(M)\times_{\mathrm{Ad}} U(\mathfrak{sp}(n,\mathbb{C}))$.

To obtain Bochner-Weitzenböck formulas for hyper-Kählerian gradients, we need curvature endomorphisms on $\mathbf{S}_\rho$. First, we see the curvature R_T on $T(M)$ defined by $R_T(V,W) := \nabla^2_{V,W} - \nabla^2_{W,V}$. This curvature satisfies

$$R_T(IV,IW) = R_T(V,W), \quad R_T(JV,JW) = R_T(V,W), \quad R_T(KV,KW) = R_T(V,W),$$
$$R_T I = I R_T, \quad R_T J = J R_T, \quad R_T K = K R_T.$$

These equations mean that we regard R_T as an endomorphism on $\mathfrak{sp}(n)$. In particular, we have

$$g(R_T(V,W)\epsilon_\alpha, \bar\epsilon_\beta) = -\mathrm{sign}(\alpha\beta) g(R_T(V,W)\epsilon_{-\beta}, \bar\epsilon_{-\alpha}),$$
$$g(R_T(V,W)\epsilon_\alpha, \epsilon_\beta) = g(R_T(V,W)\bar\epsilon_\alpha, \bar\epsilon_\beta) = 0,$$
$$R_T(\epsilon_\alpha, \bar\epsilon_\beta) = -\mathrm{sign}(\alpha\beta) R_T(\epsilon_{-\beta}, \bar\epsilon_{-\alpha}), \quad R_T(\epsilon_\alpha,\epsilon_\beta) = R_T(\bar\epsilon_\alpha, \bar\epsilon_\beta) = 0$$

for local unitary frame $\{\epsilon_\alpha\}_\alpha$. Notice that the Ricci curvature $\sum_\alpha R_T(\epsilon_\alpha, \bar\epsilon_\alpha)$ is zero, and hence, R_T depends only on the conformal Weyl tensor.

We consider the curvature R_ρ on $\mathbf{S}_\rho$,

$$R_\rho(V,W) := \nabla^2_{V,W} - \nabla^2_{W,V} \quad \text{for vector fields } V \text{ and } W.$$

Since the covariant derivative ∇ on $\mathbf{S}_\rho$ is induced from the Levi-Civita connection, the curvature R_ρ is expressed as follows:

$$\begin{aligned} R_\rho(V,W) &= 1/2\sum_{\alpha,\beta} g(R_T(V,W)\epsilon_\alpha,\bar{\epsilon}_\beta)\pi_\rho(\epsilon_\beta\otimes\epsilon_\alpha^* - \mathrm{sign}(\alpha\beta)\epsilon_{-\alpha}\otimes\epsilon_{-\beta}^*) \\ &= 1/2\sum_{\alpha,\beta} g(R_T(V,W)\epsilon_\alpha,\bar{\epsilon}_\beta)\pi_\rho(x_{\beta\alpha}). \end{aligned}$$

Since R_ρ satisfies

$$R_\rho(\epsilon_\alpha,\bar{\epsilon}_\beta) = -\mathrm{sign}(\alpha\beta)R_\rho(\epsilon_{-\beta},\bar{\epsilon}_{-\alpha}),$$

we define a *curvature endomorphism* $\hat{R}^q_\rho$ on $\mathbf{S}_\rho$ for $q = 0, 1, \cdots$ by

$$\hat{R}^q_\rho := \sum \pi_\rho(\hat{x}^q_{\beta\alpha})R_\rho(\epsilon_\alpha,\bar{\epsilon}_\beta).$$

Proposition 5.2. *The curvature endomorphisms $\{\hat{R}^q_\rho\}_q$ are self-adjoint endomorphisms of $\mathbf{S}_\rho$, namely, $((\hat{R}^q_\rho)_x\phi,\psi) = (\phi,(\hat{R}^q_\rho)_x\psi)$ at each x in M. Moreover $\{\hat{R}^q_\rho\}_q$ satisfy that*

$$(1+(-1)^q)\hat{R}^q_\rho - \frac{1-(-1)^q}{2}\hat{R}^{q-1}_\rho - \sum_{p=0}^{q-1}(-1)^p\pi_\rho(\hat{c}_{q-1-p})\hat{R}^p_\rho = 0.$$

Proof. We can show the first claim by using the Bianchi identity for R_T. The second claim follows from the universal Bochner-Weitzenböck formulas (3.14). □

For example, we have $\hat{R}^0_\rho = 0$ and $\hat{R}^2_\rho = -n\hat{R}^1_\rho$. Thus, it is enough if we think only $\{\hat{R}^{2q}_\rho\}_q$.

Now, we are in a position to give Bochner-Weitzenböck formulas.

Theorem 5.3. *Let $\{D_\nu, \mathfrak{J}D_\nu\mathfrak{J}^{-1}, D^*_\nu, \mathfrak{J}D^*_\nu\mathfrak{J}^{-1}\}_\nu$ be hyper-Kählerian gradients on a hyper-Kähler manifold. We assume that the number of irreducible components in $\mathbf{S}_\rho\otimes T^{1,0}(M)$ is N. In other words, there are N non-trivial hyper-Kählerian gradients in $\{D_\nu \mid \nu = \pm1,\cdots,\pm n\}$. Then we have the following independent identities for the hyper-Kählerian gradients.*

1.

$$\sum_\nu D^*_\nu D_\nu = \nabla^{1,0*}\nabla^{1,0} = -\sum_\alpha \nabla^2_{\bar{\epsilon}_\alpha,\epsilon_\alpha} = 1/2\nabla^*\nabla, \tag{5.2}$$

$$\sum_\nu \mathfrak{J}D^*_\nu D_\nu\mathfrak{J}^{-1} = \nabla^{0,1*}\nabla^{0,1} = -\sum_\alpha \nabla^2_{\epsilon_\alpha,\bar{\epsilon}_\alpha} = 1/2\nabla^*\nabla, \tag{5.3}$$

where $\nabla^{1,0*}\nabla^{1,0}$ and $\nabla^{0,1*}\nabla^{0,1}$ are the $(1,0)$- and the $(0,1)$- connection Laplacian, and $\nabla^*\nabla$ is the usual connection Laplacian on $\Gamma(\mathbf{S}_\rho)$.

2. *For $q = 1,\cdots,[N/2]$,*

$$\sum_\nu\{\sum_{p=0}^{2q-1}(-1)^p\pi_\rho(\hat{c}_{2q-1-p})\hat{w}^p_\nu\}(D^*_\nu D_\nu + \mathfrak{J}D^*_\nu D_\nu\mathfrak{J}^{-1}) = 2\hat{R}^{2q}_\rho. \tag{5.4}$$

3. *For* $q=0,\cdots,[\frac{N+1}{2}]-1$,

$$\sum_\nu\{2\hat{w}_\nu^{2q}-\sum_{p=0}^{2q-1}(-1)^p\pi_\rho(\hat{c}_{2q-1-p})\hat{w}_\nu^p\}(D_\nu^*D_\nu-\mathfrak{J}D_\nu^*D_\nu\mathfrak{J}^{-1})=0, \tag{5.5}$$

$$\sum_\nu\{2\hat{w}_\nu^{2q}-\sum_{p=0}^{2q-1}(-1)^p\pi_\rho(\hat{c}_{2q-1-p})\hat{w}_\nu^p\}\mathfrak{J}D_\nu^*\mathfrak{J}^{-1}D_\nu=0, \tag{5.6}$$

$$\sum_\nu\{2\hat{w}_\nu^{2q}-\sum_{p=0}^{2q-1}(-1)^p\pi_\rho(\hat{c}_{2q-1-p})\hat{w}_\nu^p\}D_\nu^*\mathfrak{J}D_\nu\mathfrak{J}^{-1}=0. \tag{5.7}$$

Proof. First, we can prove (5.2) and (5.3) by (5.1). Here, we notice that

$$-\sum_\alpha\nabla^2_{\epsilon_\alpha,\bar{\epsilon}_\alpha}-\sum_\alpha\nabla^2_{\bar{\epsilon}_\alpha,\epsilon_\alpha}=\nabla^*\nabla,\quad \sum_\alpha\nabla^2_{\epsilon_\alpha,\bar{\epsilon}_\alpha}-\sum_\alpha\nabla^2_{\bar{\epsilon}_\alpha,\epsilon_\alpha}=\hat{R}^0_\rho=0.$$

Next, by using (3.14) and (5.1), we have

$$\begin{aligned}&\sum_\nu\hat{w}_\nu^qD_\nu^*D_\nu\\ =&-\sum_{\alpha,\beta}\pi_\rho(\hat{x}^q_{\alpha\beta})(\nabla^2_{\epsilon_\beta,\bar{\epsilon}_\alpha}+R_\rho(\bar{\epsilon}_\alpha,\epsilon_\beta))=\hat{R}^q_\rho-\sum_{\alpha,\beta}\pi_\rho(\hat{x}^q_{\alpha\beta})\nabla^2_{\epsilon_\beta,\bar{\epsilon}_\alpha}\\ =&\hat{R}^q_\rho+\sum_\nu\{(-1)^q\hat{w}_\nu^q-\frac{1-(-1)^q}{2}\hat{w}_\nu^{q-1}-\sum_{p=0}^{q-1}(-1)^p\pi_\rho(\hat{c}_{q-1-p})\hat{w}_\nu^p\}\mathfrak{J}D_\nu^*D_\nu\mathfrak{J}^{-1}.\end{aligned}$$

Put this equation between $\mathfrak{J}$ and $\mathfrak{J}^{-1}$, and we have

$$\begin{aligned}&\sum\hat{w}_\nu^q\mathfrak{J}D_\nu^*D_\nu\mathfrak{J}^{-1}\\ =&\hat{R}^q_\rho+\sum\{(-1)^q\hat{w}_\nu^q-\frac{1-(-1)^q}{2}\hat{w}_\nu^{q-1}-\sum_{p=0}^{q-1}(-1)^p\pi_\rho(\hat{c}_{q-1-p})\hat{w}_\nu^p\}D_\nu^*D_\nu.\end{aligned}$$

The above two equations induce (5.4) and (5.5). We can prove (5.6) and (5.7) similarly. The independence of (5.2)-(5.7) is proved by the same discussion as the one in Proposition 3.7. □

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