Casimir elements and Bochner identities on Riemannian manifolds

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ABSTRACT We show that the principal symbols of first order geometric differential operators on Riemannian manifolds are controlled by the enveloping algebra and higher Casimir elements of $\mathfrak{so}(n)$. Then we give all the Bochner identities for the operators explicitly.
Keywords: gradients, Bochner identities, $\mathrm{SO}(n)$-modules, Casimir elements

1 Introduction

Let M be a n-dimensional oriented Riemannian manifold with metric g, and $\mathbf{SO}(M)$ be the principal $\mathrm{SO}(n)$ bundle of the oriented orthonormal frames on M. Given an irreducible unitary representation (π_ρ, V_ρ) of $\mathrm{SO}(n)$ with highest weight ρ, we have an associated vector bundle $\mathbf{S}_\rho = \mathbf{SO}(M) \times_\rho V_\rho$ on M. Then we construct first order geometric differential operators called *gradients* on M as follows. Let ∇ be the covariant derivative on $\mathbf{S}_\rho$ induced by the Levi-Civita connection,

$$\Gamma(M, \mathbf{S}_\rho) \xrightarrow{\nabla} \Gamma(M, \mathbf{S}_\rho \otimes T^*(M)).$$

Since the tensor bundle $\mathbf{S}_\rho \otimes T^*(M)$ is also an associated vector bundle, we can decompose it into irreducible bundles with respect to $\mathrm{SO}(n)$, $\mathbf{S}_\rho \otimes T^*(M) = \sum_\lambda \mathbf{S}_\lambda$. Under this decomposition, the covariant derivative ∇ splits into the sum of differential operators, $\nabla = \sum_\lambda D^\rho_\lambda$. Thus, we have a first order differential operator D^ρ_λ given by

$$D^\rho_\lambda : \Gamma(M, \mathbf{S}_\rho) \xrightarrow{\nabla} \Gamma(M, \mathbf{S}_\rho \otimes T^*(M)) \xrightarrow{\Pi^\rho_\lambda} \Gamma(M, \mathbf{S}_\lambda),$$

where Π^ρ_λ is the orthogonal projection from $\mathbf{S}_\rho \otimes T^*(M)$ onto $\mathbf{S}_\lambda$.

Many geometric first order differential operators can be realized as gradients. For example, the exterior derivative, its adjoint operator, and the

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conformal Killing operator are gradients on Riemannian manifolds. For the spin case, the Dirac operator, the twistor operator, and the Rarita-Schwinger operator are gradients on spin manifolds.

A significant feature common to the gradients is conformal covariance. For a smooth function σ on M, we change the Riemannian metric g to $g' = e^{2\sigma} g$ conformally. Then we have a principal bundle isomorphism

$$\mathbf{SO}(M) \ni p = \{e_i\}_{i=1}^n \mapsto p' = \{e'_i = e^{-\sigma} e_i\}_{i=1}^n \in \mathbf{SO}'(M),$$

and a bundle isometry

$$\psi_\rho : \mathbf{S}_\rho = \mathbf{SO}(M) \times_\rho V_\rho \ni [p, \phi] \mapsto [p', \phi] \in \mathbf{SO}'(M) \times_\rho V_\rho = \mathbf{S}'_\rho.$$

Since the gradients are defined through Riemannian metric, the gradients change under the conformal deformation as follows (see [7]):

$$D'^\rho_\lambda = (e^{-(m(\lambda)+1)\sigma} \psi_\lambda) \circ D^\rho_\lambda \circ (e^{-m(\lambda)\sigma} \psi_\rho)^{-1} : \Gamma(M, \mathbf{S}'_\rho) \to \Gamma(M, \mathbf{S}'_\lambda).$$

Here, $m(\lambda)$ is a constant called *the conformal weight* depending on ρ and λ.

From recent research for the gradients, we know various properties and applications of them: Bochner identities [1], [8] vanishing theorems, eigenvalue estimates [3], [4], ellipticities [1], refined Kato inequalities [2], [6], spherical harmonics [5], [10], and gradients on Kähler manifolds [11]. Through these papers, we see the importance of conformal weights to study gradients.

Let us see how conformal weights are used in the author's paper [11], where the gradients on Kähler manifolds called *Kählerian gradients* are discussed. A result in [11] is to give Bochner identities or Bochner-Weitzenböck formulas for Kählerian gradients, and its method is as follows:

(i) We relate the principal symbols of Kählerian gradients with the enveloping algebra of $\mathfrak{u}(m)$ by using conformal weights. Here, m is the complex dimension of underlying manifold. (ii) We find relations among the enveloping algebra and Casimir elements which correspond to the Bochner identities. (iii) These observations allow us to give the Bochner identities.

We can expect that this method is valid for gradients on Riemannian manifolds. Indeed, in this paper, we discuss the principal symbols of the gradients and give Bochner identities on Riemannian manifolds.

In [1], using the spectral resolution on the standard sphere, Branson proved that, if there are N gradients $\{D^\rho_{\lambda_i}\}_{i=1}^N$ on associated bundle $\mathbf{S}_\rho$, then there exist $[N/2]$ independent Bochner identities. More precisely, the space of $(b_{\lambda_1}, \cdots, b_{\lambda_N}) \in \mathbb{R}^N$ such that

$$\sum_{1 \le i \le N} b_{\lambda_i} (D^\rho_{\lambda_i})^* D^\rho_{\lambda_i} = (\text{curvature action}) \tag{1.1}$$

has dimension $[N/2]$ in $\mathbb{R}^N$, and the vector $(b_{\lambda_1}, \cdots, b_{\lambda_N})$ satisfies the following linear simultaneous equations:

$$\sum_{1 \le i \le N} b_{\lambda_i} \tilde{c}_{\rho\lambda_i} (m(\lambda_i) - \frac{n-1}{2})^{2j} = 0, \quad j = 0, \cdots, [(N+1)/2] - 1,$$

where $\tilde{c}_{\rho\lambda_i}$ is a constant depending on ρ and λ_i. Note that, in the same paper [1], Branson also answered ellipticity problem. Thus, the Bochner identities on Riemannian manifolds have been already known. But, our method is quite different to Branson's one, and relevant to the paper [6] by Calderbank, Gauduchon and Herzlich. We deal with the principal symbols directly and find out that their structure is controlled by the enveloping algebra and Casimir elements of $\mathfrak{so}(n)$. This observation leads us to explicit formulas of the Bochner identities for the gradients.

This paper is organized as follows. In section 2, we study the enveloping algebra and Casimir elements of $\mathfrak{so}(n)$, and give relations among them corresponding to Bochner identities. In section 3, we define the principal symbols of gradients called *the Clifford homomorphisms*. We relate them with the enveloping algebra by using conformal weights. As a corollary, we calculate the eigenvalues of Casimir elements. In the last section, we give all the Bochner identities for the gradients explicitly. In other words, we give explicit formulas of the vector $(b_{\lambda_1}, \cdots, b_{\lambda_N})$ and curvature action in (1.1).

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2 Enveloping algebra and Casimir elements

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the standard basis $\{e_i\}_{i=1}^n$, and $\mathfrak{so}(n)$ be the Lie algebra of the spin group $\mathrm{Spin}(n)$ or the special orthogonal group $\mathrm{SO}(n)$. It is known that there is a natural isomorphism between $\bigwedge^2(\mathbb{R}^n)$ and $\mathfrak{so}(n)$ by associating $e_i \wedge e_j$ with a skew-symmetric endomorphism

$$(e_i \wedge e_j)(v) = \langle e_i, v\rangle e_j - \langle e_j, v\rangle e_i \quad \text{for any } v \in \mathbb{R}^n.$$

The basis of $\mathfrak{so}(n)$ consists of $\{e_i \wedge e_j\}_{i<j}$. For $i, j = 1, \cdots, n$, we put $e_{ij} := e_i \wedge e_j$ and have relations

$$e_{ij} = -e_{ji},$$
$$[e_{kl}, e_{ij}] = \delta_{ki} e_{lj} + \delta_{kj} e_{il} - \delta_{il} e_{kj} - \delta_{lj} e_{ik}$$

in $\mathfrak{so}(n)$.

We denote the complexification of $\mathfrak{so}(n)$ by $\mathfrak{so}(n,\mathbb{C})$ and its enveloping algebra by $U(\mathfrak{so}(n,\mathbb{C}))$. The center $\mathfrak{Z}$ of the enveloping algebra is characterize as the invariant subalgebra in $U(\mathfrak{so}(n,\mathbb{C}))$ under adjoint action of $\mathrm{SO}(n)$ or $\mathfrak{so}(n)$, that is, $\mathfrak{Z}=U(\mathfrak{so}(n,\mathbb{C}))^{\mathrm{SO}(n)}$ (see [13]). We call elements in $\mathfrak{Z}$ *Casimir elements.* It follows from Schur's lemma that a Casimir element is a multiple of the identity on any irreducible $\mathfrak{so}(n)$-module.

We shall define some elements in $U(\mathfrak{so}(n,\mathbb{C}))$ whose traces are Casimir elements. For any non-negative integer q, we define an element e_{ij}^q with degree q in $U(\mathfrak{so}(n,\mathbb{C}))$ by

$$e_{ij}^q := \begin{cases} \sum_{1\le i_1,\cdots,i_{q-1}\le n} e_{ii_1}e_{i_1i_2}\cdots e_{i_{q-1}j} & q\ge 1 \\ \delta_{ij} & q=0. \end{cases} \tag{2.1}$$

This e_{ij}^q behaves like e_{ij} under adjoint action of $\mathfrak{so}(n)$.

Lemma 1. *The elements* $\{e_{ij}^q | q\in\mathbb{Z}_{\ge 0}, i,j=1,\cdots,n\}$ *satisfy that*

$$[e_{kl},e_{ij}^q] = \delta_{ki}e_{lj}^q + \delta_{kj}e_{il}^q - \delta_{il}e_{kj}^q - \delta_{lj}e_{ik}^q, \tag{2.2}$$

$$\sum_{1\le k\le n} e_{ik}^p e_{kj}^q = e_{ij}^{p+q}. \tag{2.3}$$

Proof. The second equation follows from the definition (2.1). To prove the first equation, we consider the adjoint action of $\mathfrak{so}(n)$ on e_{ij}^q. Then we have

$$\begin{aligned} &[e_{kl},e_{ij}^q] \\ =&\sum_{i_1,\cdots,i_{q-1}} [e_{kl},e_{ii_1}]e_{i_1i_2}\cdots e_{i_{q-1}j} + \cdots + \sum_{i_1,\cdots,i_{q-1}} e_{ii_1}e_{i_1i_2}\cdots [e_{kl},e_{i_{q-1}j}] \\ =&\delta_{ki}e_{lj}^q + \delta_{kj}e_{il}^q - \delta_{il}e_{kj}^q - \delta_{lj}e_{ik}^q. \end{aligned}$$

Thus, we have proved the lemma. □

It follows from (2.2) that the trace $\sum_i e_{ii}^q$ is invariant under adjoint action of $\mathfrak{so}(n)$.

Corollary 1. *The trace of* e_{ij}^q *is in the center* $\mathfrak{Z}$ *of* $U(\mathfrak{so}(n,\mathbb{C}))$. *Thus, we have the Casimir element* $c_q := \sum_{i=1}^n e_{ii}^q$ *with degree* q *for* $q=0,1,\cdots$.

Remark 1. *We remark that* c_0 *is equal to* n, c_1 *is equal to* 0, *and* $c_2=\sum_{ij} e_{ij}e_{ji}$ *is the usual Casimir element.*

In the case of $n=2m$, we have another Casimir element pf given by

$$pf := \sum_{\sigma\in\mathfrak{S}_{2m}} \mathrm{sign}(\sigma) e_{\sigma(1)\sigma(2)}e_{\sigma(3)\sigma(4)}\cdots e_{\sigma(2m-1)\sigma(2m)}. \tag{2.4}$$

We call this Casimir element *the Pfaffian element.* From [12] or [13], we know the following facts on these Casimir elements:

1. In the case of $n = 2m$, $\{c_2, c_4, \cdots, c_{2m-2}, pf\}$ generate the center $\mathfrak{Z}$ algebraically.
2. In the case of $n = 2m+1$, $\{c_2, c_4, \cdots, c_{2m}\}$ generate the center $\mathfrak{Z}$ algebraically.

Now, the purpose in this section is to give algebraic identities for $\{e_{ij}^q\}_q$, which will correspond to the Bochner identities. We can show by an induction that e_{ij}^q is a linear combination of $\{e_{ji}^p\}_{p=0}^q$ whose coefficients are Casimir elements,

$$e_{ij}^q = \sum_{p=0}^{q} a_{q,p} e_{ji}^p, \quad a_{q,p} \in \mathfrak{Z}.$$

Here, we can obtain a recursion formula for $\{a_{q,p}\}_{q \geq p \geq 0}$ by using (2.2) and (2.3). Since it is complicated to solve the recursion formula, we have to translate e_{ij}^q to another element $\hat{e}_{ij}^q$. We define a translated element $\hat{e}_{ij}$ by

$$\hat{e}_{ij} := e_{ij} + \frac{n-1}{2}\delta_{ij},$$

and define $\hat{e}_{ij}^q$ with degree q by

$$\hat{e}_{ij}^q := \begin{cases} \sum_{1 \leq i_1, \cdots, i_{q-1} \leq n} \hat{e}_{ii_1} \hat{e}_{i_1 i_2} \cdots \hat{e}_{i_{q-1} j} & q \geq 1 \\ \delta_{ij} & q = 0. \end{cases}$$

Note that $\hat{e}_{ij}^q$ is related to e_{ij}^q with the equation

$$\hat{e}_{ij}^q = \sum_{p=0}^{q} \binom{q}{p} \left(\frac{n-1}{2}\right)^{q-p} e_{ij}^p.$$

We can easily show the following relations for $\{\hat{e}_{ij}^q\}_q$ in the same way as lemma 1.

Lemma 2. *The translated elements* $\{\hat{e}_{ij}^q | q \in \mathbb{Z}_{\geq 0}, i, j = 1, \cdots, n\}$ *satisfy that*

$$[\hat{e}_{kl}, \hat{e}_{ij}^q] = \delta_{ki}\hat{e}_{lj}^q + \delta_{kj}\hat{e}_{il}^q - \delta_{il}\hat{e}_{kj}^q - \delta_{lj}\hat{e}_{ik}^q, \tag{2.5}$$

$$\sum_k \hat{e}_{ik}^p \hat{e}_{kj}^q = \hat{e}_{ij}^{p+q}, \tag{2.6}$$

$$\hat{e}_{ij} = -\hat{e}_{ji} + (n-1)\delta_{ij}. \tag{2.7}$$

By using this lemma, we have algebraic identities for $\hat{e}_{ij}^q$ and $\hat{c}_q := \sum_i \hat{e}_{ii}^q$.

Theorem 1. *The translated element* $\hat{e}_{ij}^q$ *is a linear combination of* $\{\hat{e}_{ji}^p\}_{p=0}^q$ *whose coefficients are Casimir elements,*

$$\hat{e}_{ij}^q = (-1)^q \hat{e}_{ji}^q - \frac{1-(-1)^q}{2} \hat{e}_{ji}^{q-1} + \sum_{p=0}^{q-1} (-1)^p \hat{c}_{q-1-p} \hat{e}_{ji}^p. \tag{2.8}$$

Here, $\hat{c}_q := \sum_i \hat{e}_{ii}^q$ *is in the center* $\mathfrak{Z}$*. Thus, we have*

$$\hat{e}_{ij}^{2q} = \hat{e}_{ji}^{2q} + \sum_{p=0}^{2q-1} (-1)^p \hat{c}_{2q-1-p} \hat{e}_{ji}^p, \tag{2.9}$$

$$\hat{e}_{ij}^{2q+1} = -\hat{e}_{ji}^{2q+1} - \hat{e}_{ji}^{2q} + \sum_{p=0}^{2q} (-1)^p \hat{c}_{2q-p} \hat{e}_{ji}^p. \tag{2.10}$$

Proof. From (2.5)-(2.7), we have

$$\begin{aligned} \hat{e}_{ij}^{q+1} &= \sum_k [\hat{e}_{ik}, \hat{e}_{kj}^q] + \sum_k \hat{e}_{kj}^q \hat{e}_{ik} \\ &= \sum_k (\delta_{ik} \hat{e}_{kj}^q + \delta_{ij} \hat{e}_{kk}^q - \delta_{kk} \hat{e}_{ij}^q - \delta_{kj} \hat{e}_{ki}^q) + \sum_k \hat{e}_{kj}^q (-\hat{e}_{ki} + (n-1)\delta_{ki}) \\ &= \delta_{ji} \hat{c}_q - \hat{e}_{ji}^q - \sum_{k=1}^n \hat{e}_{kj}^q \hat{e}_{ki}. \end{aligned} \tag{2.11}$$

Setting $\hat{e}_{ij}^q = \sum_{p=0}^q \hat{a}_{q,p} \hat{e}_{ji}^p$, we shall produce a recursion formula for $\{\hat{a}_{q,p}\}_{q,p}$, where each $\hat{a}_{q,p}$ is in the center $\mathfrak{Z}$. It follows from (2.11) that

$$\begin{aligned} \hat{e}_{ij}^{q+1} &= \delta_{ji} \hat{c}_q - \hat{e}_{ji}^q - \sum_k \hat{e}_{kj}^q \hat{e}_{ki} \\ &= \delta_{ji} \hat{c}_q - \hat{e}_{ji}^q - \sum_k \sum_p \hat{a}_{q,p} \hat{e}_{jk}^p \hat{e}_{ki} \\ &= \delta_{ji} \hat{c}_q - \hat{e}_{ji}^q - \sum_{p=0}^q \hat{a}_{q,p} \hat{e}_{ji}^{p+1} \\ &= -\hat{a}_{q,q} \hat{e}_{ji}^{q+1} + (-\hat{a}_{q,q-1} - 1) \hat{e}_{ji}^q - \sum_{p=0}^{q-2} \hat{a}_{q,p} \hat{e}_{ji}^{p+1} + \hat{c}_q \delta_{ji} \\ &= \sum_{p=0}^{q+1} \hat{a}_{q+1,p} \hat{e}_{ji}^p. \end{aligned}$$

Then we have a recursion formula for $\{\hat{a}_{q,p}\}_{q\geq p\geq 0}$,

$$\hat{a}_{q+1,p} = \begin{cases} -\hat{a}_{q,q}, & p = q+1, \\ -\hat{a}_{q,q-1} - 1, & p = q, \\ -\hat{a}_{q,p-1}, & 1 \leq p \leq q-1, \\ \hat{c}_q, & p = 0. \end{cases}$$

We can easily solve it and have

$$\hat{a}_{q+1,p} = \begin{cases} (-1)^{q+1}, & p = q+1, \\ (-1)^q(n-1) - \frac{1-(-1)^q}{2} = (-1)^q \hat{c}_0 - \frac{1-(-1)^{q+1}}{2}, & p = q, \\ (-1)^p \hat{c}_{q-p}, & 0 \leq p \leq q-1. \end{cases}$$

Thus, we have proved the theorem. □

Take the trace on the equation (2.8), and we have identities for the Casimir elements $\{\hat{c}_q\}_q$. The trace of (2.9) is a trivial identity, while the trace of (2.10) gives the following corollary.

Corollary 2. *The Casimir elements* $\{\hat{c}_0, \hat{c}_1, \hat{c}_2, \cdots\}$ *satisfy*

$$2\hat{c}_{2q+1} = -\hat{c}_{2q} + \sum_{p=0}^{2q} (-1)^p \hat{c}_{2q-p}\hat{c}_p \tag{2.12}$$

for $q = 0, 1, \cdots$ *. In particular, the Casimir element* $\hat{c}_{2q+1}$ *with odd degree can be represented as a polynomial of* $\{\hat{c}_0, \hat{c}_2, \cdots, \hat{c}_{2q}\}$.

3 The Clifford homomorphisms: the principal symbols of gradients

In this section, we shall discuss the Clifford homomorphisms introduced by the author as a generalization of the Clifford multiplication in [9] and [11].

Let (π_ρ, V_ρ) be an irreducible unitary representation of the Lie algebra $\mathfrak{so}(n)$ with highest weight ρ. The highest weight ρ is a vector $\rho = (\rho^1, \cdots, \rho^m)$ in $\mathbb{Z}^m \cup (\mathbb{Z}+1/2)^m$ satisfying the dominant condition (see [13]):

$$\rho^1 \geq \rho^2 \geq \cdots \geq \rho^{m-1} \geq |\rho^m|, \quad n = 2m,$$
$$\rho^1 \geq \rho^2 \geq \cdots \geq \rho^m \geq 0, \quad n = 2m+1.$$

Conversely, every dominant integral or half-integral weight is the highest weight of an irreducible unitary representation. We denote the standard basis of $\mathbb{Z}^m$ by $\{\mu_i\}_{i=1}^m$,

$$\mu_i = (\underbrace{0, \cdots, 0}_{i-1}, 1, \underbrace{0, \cdots, 0}_{m-i}).$$

With this notation, (π_{μ_1}, V_{μ_1}) is the natural representation of $\mathfrak{so}(n)$ on $\mathbb{R}^n \otimes \mathbb{C}$.

Now, we consider the tensor representation $(\pi_\rho \otimes \pi_{\mu_1}, V_\rho \otimes_{\mathbb{C}} \mathbb{R}^n)$ and its irreducible decomposition

$$V_\rho \otimes \mathbb{R}^n = \sum_\lambda V_\lambda,$$

where we adopt the inner product on V_λ induced by the tensor inner product on $V_\rho \otimes \mathbb{R}^n$.

The highest weights of irreducible components are characterized as follows (see [7]):

1. When $n = 2m$ or when $n = 2m+1$ and $\rho^m = 0$, $\lambda = \rho \pm \mu_i$ for $i = 1, \cdots, m$ such that λ is dominant.

2. When $n = 2m+1$ and $\rho^m > 0$, $\lambda = \rho$ or $\lambda = \rho \pm \mu_i$ for $i = 1, \cdots, m$ such that λ is dominant.

In particular, the irreducible components occur with multiplicity one.

The Clifford homomorphism is defined through the orthogonal projection from $V_\rho \otimes \mathbb{R}^n$ onto each irreducible component.

Definition 1. *For ξ in $\mathbb{R}^n$, we define a linear mapping $p^\rho_\lambda(\xi)$ from V_ρ to V_λ by*

$$\mathbb{R}^n \times V_\rho \ni (\xi, \phi) \mapsto p^\rho_\lambda(\xi)\phi := \Pi^\rho_\lambda(\phi \otimes \xi) \in V_\lambda,$$

where Π^ρ_λ is the orthogonal projection from $V_\rho \otimes \mathbb{R}^n$ onto irreducible component V_λ. We denote by $p^\rho_\lambda(\xi)^*$ the adjoint of $p^\rho_\lambda(\xi)$ with respect to the inner products on V_ρ and V_λ. We call these linear mappings $p^\rho_\lambda(\xi)$ and $p^\rho_\lambda(\xi)^*$ the Clifford homomorphisms associated to ρ and λ.

Example 1. *Let $(\pi_{\Delta^\pm}, V_{\Delta^\pm})$ be the spinor representation with highest weight $\Delta^\pm = (1/2, \cdots, \pm 1/2)$. There are two components $V_{\Delta^\mp}$ and $V_{T^\pm}$ in $V_{\Delta^\pm} \otimes \mathbb{R}^n$, where $T^\pm$ is $(3/2, 1/2, \cdots, \pm 1/2)$. The Clifford homomorphism $p^{\Delta^\pm}_{\Delta^\mp}(u) : V_{\Delta^\pm} \to V_{\Delta^\mp}$ is the usual Clifford multiplication on spinor space up to a normalization.*

The next proposition says that the Clifford homomorphisms are related to the enveloping algebra.

Proposition 1 ([6], [9]). *Let $\{e_i\}_{i=1}^n$ be the standard orthonormal basis of $\mathbb{R}^n$. The Clifford homomorphisms $\{p^\rho_\lambda\}_\lambda$ satisfy*

$$\sum_\lambda (-m(\lambda))^q p^\rho_\lambda(e_i)^* p^\rho_\lambda(e_j) = \pi_\rho(e^q_{ij}) \tag{3.1}$$

for $q = 0, 1, \cdots,$ *and* $i, j = 1, \cdots, n$. *The constant* $m(\lambda)$ *is called the conformal weight associated to* ρ *and* λ *defined by*

$$m(\lambda) = \frac{1}{2}(n - \|\delta + \lambda\|^2 + \|\delta + \rho\|^2 - 1),$$

where δ *is half the sum of the positive roots of* $\mathfrak{so}(n)$, *and* $\|\cdot\|$ *is the standard norm on the space of weights. In particular, we have*

$$\sum_\lambda (-m(\lambda))^q \sum_i p^\rho_\lambda(e_i)^* p^\rho_\lambda(e_i) = \pi_\rho(c_q).$$

Remark 2. *We can easily show that the conformal weights associated to irreducible components differ from each other except the following case (see [6]). When* $n = 2m$, $\rho^{m-1} > 0$, *and* $\rho^m = 0$, *we have irreducible components* $V_{\rho+\mu_m}$ *and* $V_{\rho-\mu_m}$ *in* $V_\rho \otimes \mathbb{R}^n$ *whose conformal weights coincide,* $m(\rho + \mu_m) = m(\rho - \mu_m)$. *We call this case the exceptional case.*

Proof. First, we shall prove (3.1) for $q = 0$. For ϕ and ψ in V_ρ, we have

$$\langle \phi \otimes e_i, \psi \otimes e_j \rangle = \langle \phi, \psi \rangle \langle e_i, e_j \rangle = \delta_{ij} \langle \phi, \psi \rangle.$$

On the other hand, we show from the definition of the Clifford homomorphisms that

$$\langle \phi \otimes e_i, \psi \otimes e_j \rangle = \sum_\lambda \langle p^\rho_\lambda(e_i)\phi, p^\rho_\lambda(e_j)\psi \rangle = \langle \sum_\lambda p^\rho_\lambda(e_j)^* p^\rho_\lambda(e_i)\phi, \psi \rangle.$$

Since the above two equations are valid for any ϕ and ψ, we have

$$\sum_\lambda p^\rho_\lambda(e_j)^* p^\rho_\lambda(e_i) = \delta_{ji}. \tag{3.2}$$

In particular, for ξ and η in $\mathbb{R}^n$,

$$\sum_\lambda p^\rho_\lambda(\xi)^* p^\rho_\lambda(\eta) = \langle \xi, \eta \rangle.$$

Next, to prove (3.1) for $q \geq 1$, we need an operator C on $V_\rho \otimes \mathbb{R}^n$ defined by

$$C := \pi_\rho \otimes \pi_{\mu_1}(c_2) - \pi_\rho(c_2) \otimes \mathrm{id} - \mathrm{id} \otimes \pi_{\mu_1}(c_2).$$

Here, c_2 is the usual Casimir element $c_2 = \sum_{ij} e_{ij} e_{ji}$, which is $2(\|\delta+\rho\|^2 - \|\delta\|^2)\mathrm{id}$ on irreducible $\mathfrak{so}(n)$-module V_ρ (see [7], [8]). Then the operator C is $-4m(\lambda)\mathrm{id}$ on irreducible component V_λ of $V_\rho \otimes \mathbb{R}^n$. On the other hand, we can show that the operator C is realized by

$$C = 2 \sum_{ij} \pi_\rho(e_{ij}) \otimes \pi_{\mu_1}(e_{ji}).$$

So, for $\phi \otimes e_i$ in $V_\rho \otimes \mathbb{R}^n$, we have

$$\begin{aligned} C(\phi \otimes e_i) &= 2\sum_{kl} \pi_\rho(e_{kl})\phi \otimes \pi_{\mu_1}(e_{lk})e_i = 2\sum_{kl} \pi_\rho(e_{kl})\phi \otimes (\delta_{il}e_k - \delta_{ki}e_l) \\ &= 4\sum_k \pi_\rho(e_{ki})\phi \otimes e_k = 4\sum_\lambda \sum_k p^\rho_\lambda(e_k)\pi_\rho(e_{ki})\phi \end{aligned}$$

As a result, we have $\sum_k p^\rho_\lambda(e_k)\pi_\rho(e_{ki}) = -m(\lambda)p^\rho_\lambda(e_i)$ for any λ. This equation and (3.2) induce (3.1) for $q \geq 1$. □

If we translate $m(\lambda)$ to $\hat{m}(\lambda) := m(\lambda) - \frac{n-1}{2}$, then we can similarly show that

$$\sum_\lambda (-\hat{m}(\lambda))^q p^\rho_\lambda(e_i)^* p^\rho_\lambda(e_j) = \pi_\rho(\hat{e}^q_{ij}). \tag{3.3}$$

Since we have known the relation (2.8) for $\hat{e}^q_{ij}$, we obtain algebraic relations for the Clifford homomorphisms.

Theorem 2. *Let $\hat{m}(\lambda) = m(\lambda) - \frac{n-1}{2}$ be the translated conformal weight associated to ρ and λ. Then there are the following algebraic relations for the Clifford homomorphisms: for ξ and η in $\mathbb{R}^n$,*

$$\begin{aligned} &\sum_\lambda \hat{m}(\lambda)^{2q} p^\rho_\lambda(\xi)^* p^\rho_\lambda(\eta) \\ &= \sum_\lambda \left\{ \hat{m}(\lambda)^{2q} + \sum_{p=0}^{2q-1} \hat{m}(\lambda)^p \pi_\rho(\hat{c}_{2q-1-p}) \right\} p^\rho_\lambda(\eta)^* p^\rho_\lambda(\xi), \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\sum_\lambda \hat{m}(\lambda)^{2q+1} p^\rho_\lambda(\xi)^* p^\rho_\lambda(\eta) \\ &= \sum_\lambda \left\{ -\hat{m}(\lambda)^{2q+1} + \hat{m}(\lambda)^{2q} - \sum_{p=0}^{2q} \hat{m}(\lambda)^p \pi_\rho(\hat{c}_{2q-p}) \right\} p^\rho_\lambda(\eta)^* p^\rho_\lambda(\xi), \end{aligned} \tag{3.5}$$

and

$$\sum_\lambda p^\rho_\lambda(\xi)^* p^\rho_\lambda(\eta) = \langle \xi, \eta \rangle. \tag{3.6}$$

Example 2. *On spinor space, these relation give the usual Clifford relation $\xi \cdot \eta \cdot + \eta \cdot \xi \cdot = -2\langle \xi, \eta \rangle$.*

For $n = 2m$, we get another identity through the Pfaffian element pf in (2.4). Considering the action pf on $V_\rho \otimes \mathbb{R}^n$, we can relate the Pfaffian element pf to the Clifford homomorphisms $\{p^\rho_\lambda\}_\lambda$ as follows.

Proposition 2 ([9]). *The Clifford homomorphisms* $\{p^\rho_\lambda\}_\lambda$ *satisfy that*

$$\sum \pi_\lambda(pf)p^\rho_\lambda(e_j)^* p^\rho_\lambda(e_i) = \delta_{ij}\pi_\rho(pf) +$$
$$2m(1-\delta_{ij})\mathrm{sign}\begin{pmatrix} 1 & 2 & 3 & \cdots & 2m \\ i & j & 1 & \cdots & 2m \end{pmatrix} \sum_{\sigma\in\mathfrak{S}^{i,j}_{2m}} \mathrm{sign}(\sigma)\pi_\rho(e_{\sigma(1)\sigma(2)})\cdots\pi_\rho(e_{\sigma(2m-1)\sigma(2m)}),$$

where $\mathfrak{S}^{i,j}_{2m}$ *is the permutation group of* $\{1,\cdots,2m\}\setminus\{i,j\}$*. In particular, we have*

$$\sum_\lambda \pi_\lambda(pf)(p^\rho_\lambda(\xi)^* p^\rho_\lambda(\eta) + p^\rho_\lambda(\eta)^* p^\rho_\lambda(\xi)) = 2\pi_\rho(pf)\langle\xi,\eta\rangle \qquad (3.7)$$

for ξ *and* η *in* $\mathbb{R}^n$*.*

In the rest of this section, we calculate the constant $\pi_\rho(\hat{c}_q)$, that is, the eigenvalues of $\hat{c}_q$. Here, our method is based on [6] or [12]. First, we need the following lemma.

Lemma 3. *The orthogonal projection* $\Pi^\rho_\lambda : V_\rho\otimes\mathbb{R}^n \to V_\lambda \subset V_\rho\otimes\mathbb{R}^n$ *is realized as follows:*

$$\Pi^\rho_\lambda(\phi\otimes u) = \sum_i p^\rho_\lambda(e_i)^* p^\rho_\lambda(u)\phi\otimes e_i. \qquad (3.8)$$

Proof. We can easily show that the Clifford homomorphisms are compatible with the action of $\mathrm{SO}(n)$,

$$p^\rho_\lambda(\pi_{\mu_1}(g)u) = \pi_\lambda(g)p^\rho_\lambda(u)\pi_\rho(g^{-1})$$

for g in $\mathrm{SO}(n)$ and u in $\mathbb{R}^n$. It follows that the following mapping is a $\mathfrak{so}(n)$-equivariant injection,

$$\Phi_\lambda : V_\lambda \ni \psi \mapsto \sum_i p^\rho_\lambda(e_i)^*\psi\otimes e_i \in V_\rho\otimes\mathbb{R}^n.$$

Then we decompose $\phi\otimes u$,

$$\begin{aligned}\phi\otimes u = \sum_i\langle e_i,u\rangle\phi\otimes e_i &= \sum_i\sum_\lambda p^\rho_\lambda(e_i)^* p^\rho_\lambda(u)\phi\otimes e_i \\ &= \sum_\lambda\sum_i p^\rho_\lambda(e_i)^* p^\rho_\lambda(u)\phi\otimes e_i,\end{aligned}$$

and have the projection formula (3.8). □

Proposition 3. *We set* $d(\rho) := \dim V_\rho$*. The Clifford homomorphism* p^ρ_λ *satisfies*

$$\sum_i p^\rho_\lambda(e_i)^* p^\rho_\lambda(e_i) = d(\lambda)/d(\rho).$$

Hence, eigenvalues of c_q *and* $\hat{c}_q$ *on irreducible* $\mathfrak{so}(n)$ *-module* V_ρ *are*

$$\pi_\rho(c_q) = \frac{1}{d(\rho)}\sum_\lambda (-m(\lambda))^q d(\lambda), \quad \pi_\rho(\hat{c}_q) = \frac{1}{d(\rho)}\sum_\lambda (-\hat{m}(\lambda))^q d(\lambda).$$

Moreover, we have a relation for eigenvalues of the Pfaffian element,

$$m\pi_\rho(pf) = \frac{1}{d(\rho)}\sum_\lambda \pi_\lambda(pf)d(\lambda).$$

Proof. Let $\{\phi_s\}_{s=1}^{\dim V_\rho}$ be an orthonormal basis of V_ρ. Taking the trace of Π^ρ_λ, we have

$$\begin{aligned} d(\lambda) =& \sum_{s,i}(\Pi^\rho_\lambda(\phi_s\otimes e_i), \phi_s\otimes e_i) = \sum(\sum_j p^\rho_\lambda(e_j)^* p^\rho_\lambda(e_i)(\phi_s)\otimes e_j, \phi_s\otimes e_i) \\ =& \sum(\sum_j p^\rho_\lambda(e_j)^* p^\rho_\lambda(e_i)(\phi_s), \phi_s)\delta_{ij} = \sum_s(\phi_s,\phi_s)\sum_i p^\rho_\lambda(e_i)^* p^\rho_\lambda(e_i) \\ =& d(\rho)\sum_i p^\rho_\lambda(e_i)^* p^\rho_\lambda(e_i). \end{aligned}$$

□

Remark 3. *In [12] or [13], we know that the eigenvalue of the Pfaffian element on irreducible* $\mathfrak{so}(n)$ *-module* V_ρ *is*

$$\pi_\rho(pf) = (\sqrt{-1})^m 2^m m!(\rho^1+m-1)(\rho^2+m-2)\cdots(\rho^{m-1}+1)\rho^m. \quad (3.9)$$

4 The Bochner identities for the gradients

In this section, we give explicit formulas of all the Bochner identities for gradients on Riemannian or spin manifolds. We consider only the gradients on Riemannian manifolds. Since the discussions in this section are valid for the gradients on spin manifolds, the spin case is left to the readers. Let M be an oriented Riemannian manifold, and $\mathbf{SO}(M)$ be the principal $\mathrm{SO}(n)$ bundle of the oriented orthonormal frames on M. For an irreducible unitary representation (π_ρ, V_ρ) of $\mathrm{SO}(n)$, we have an associated vector bundle $\mathbf{S}_\rho := \mathbf{SO}(M)\times_\rho V_\rho$. The gradients are first order differential operators realized as irreducible components of the covariant derivative ∇ as follows:

$$D^\rho_\lambda : \Gamma(M,\mathbf{S}_\rho) \xrightarrow{\nabla} \Gamma(M,\mathbf{S}_\rho\otimes T^*(M)) \xrightarrow{\Pi^\rho_\lambda} \Gamma(M,\mathbf{S}_\lambda).$$

Here, the covariant derivative ∇ is the one induced by the Levi-Civita connection and Π^ρ_λ is the orthogonal projection defined fiberwise. We call this first order differential operator D^ρ_λ *the gradient associated to* ρ *and* λ.

We have already known the principal symbol of the gradient D^ρ_λ, that is, the Clifford homomorphism associated to ρ and λ. So there is a formula for the gradient D^ρ_λ like the Dirac operator,

$$D^\rho_\lambda = \sum_i p^\rho_\lambda(e_i)\nabla_{e_i} : \Gamma(M, \mathbf{S}_\rho) \to \Gamma(M, \mathbf{S}_\lambda),$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame. Furthermore, we have a formula of the formal adjoint operator of D^ρ_λ,

$$(D^\rho_\lambda)^* = \sum_i -p^\rho_\lambda(e_i)^*\nabla_{e_i} : \Gamma(M, \mathbf{S}_\lambda) \to \Gamma(M, \mathbf{S}_\rho).$$

The algebraic identities (3.4)-(3.6) allow us to have the Bochner identities for the gradients $\{D^\rho_\lambda\}_\lambda$. But, because of non-commutativity of ∇, we need the following curvature endomorphisms on $\mathbf{S}_\rho$.

Definition 2. *Let $R_\rho(X,Y)$ be the curvature of ∇ on $\mathbf{S}_\rho$ for vector fields X and Y, and $\{e_i\}_{i=1}^n$ be a local orthonormal frame of M. We define the curvature endomorphism $\hat{R}^q_\rho$ for $q = 1, 2, \cdots$ by*

$$\hat{R}^q_\rho := \sum_{i,j} \pi_\rho(\hat{e}^q_{ij}) R_\rho(e_i, e_j) \quad \in \Gamma(M, \mathrm{End}(\mathbf{S}_\rho)).$$

Now, we are in a position to obtain the Bochner identities.

Theorem 3 (The Bochner identities for the gradients). *Let $\{D^\rho_\lambda\}_\lambda$ be the gradients on $\Gamma(M, \mathbf{S}_\rho)$, and $\{(D^\rho_\lambda)^*\}_\lambda$ be their formal adjoints. There exist the following identities for the gradients:*

$$\sum_\lambda \{\sum_{p=0}^{2q-1} \pi_\rho(\hat{c}_{2q-1-p})\hat{m}(\lambda)^p\}(D^\rho_\lambda)^* D^\rho_\lambda = \hat{R}^{2q}_\rho \tag{4.1}$$

for $q = 1, 2, \cdots$, and

$$\sum_\lambda (D^\rho_\lambda)^* D^\rho_\lambda = \nabla^*\nabla. \tag{4.2}$$

Here, $\hat{R}^q_\rho$ is the curvature endomorphism and $\nabla^*\nabla$ is the connection Laplacian on $\mathbf{S}_\rho$. In the exceptional case such that $n = 2m$, $\rho^{m-1} > 0$, and $\rho^m = 0$, there exists another identity

$$\begin{aligned} &2((D^\rho_{\rho+\mu_m})^* D^\rho_{\rho+\mu_m} - (D^\rho_{\rho-\mu_m})^* D^\rho_{\rho-\mu_m}) \\ &= -\sum_{i,j} (p^\rho_{\rho+\mu_m}(e_i)^* p^\rho_{\rho+\mu_m}(e_j) - p^\rho_{\rho-\mu_m}(e_i)^* p^\rho_{\rho-\mu_m}(e_j)) R_\rho(e_i, e_j). \end{aligned} \tag{4.3}$$

Proof. We define a second order differential operator $\nabla^2_{X,Y}$ on $\Gamma(M, \mathbf{S}_\rho)$ for vector fields X and Y by $\nabla^2_{X,Y} := \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$. It follows from (3.3) that

$$\sum_\lambda (-\hat{m}(\lambda))^q (D^\rho_\lambda)^* D^\rho_\lambda = -\sum_{ij} \pi_\rho(\hat{e}^q_{ij}) \nabla^2_{e_i,e_j}.$$

In particular, when q is zero, we have (4.2).

To give the identities (4.1), we use the algebraic relation (2.9) for the enveloping algebra. Then,

$$\begin{aligned}\hat{R}^{2q}_\rho &= \sum_{i,j} \pi_\rho(\hat{e}^{2q}_{ij})(\nabla^2_{e_i,e_j} - \nabla^2_{e_j,e_i}) \\ &= -\sum_\lambda \hat{m}(\lambda)^{2q} (D^\rho_\lambda)^* D^\rho_\lambda - \sum_{i,j} \pi_\rho(\hat{e}^{2q}_{ji} + \sum_p (-1)^p \hat{c}_{2q-1-p} \hat{e}^p_{ji}) \nabla^2_{e_j,e_i} \\ &= \sum_\lambda \{ \sum_{p=0}^{2q-1} \pi_\rho(\hat{c}_{2q-1-p}) \hat{m}(\lambda)^p \} (D^\rho_\lambda)^* D^\rho_\lambda.\end{aligned}$$

In the exceptional case, we show from (3.9) that the eigenvalue $\pi_\lambda(pf)$ on V_λ is zero except $\lambda = \rho \pm \mu_m$. Furthermore, we know $\pi_\rho(pf) = 0$ and $\pi_{\rho+\mu_m}(pf) = -\pi_{\rho-\mu_m}(pf)$. So, the relation (3.7) for the exceptional case becomes

$$\begin{aligned}&(p^\rho_{\rho+\mu_m}(e_j)^* p^\rho_{\rho+\mu_m}(e_i) + p^\rho_{\rho+\mu_m}(e_i)^* p^\rho_{\rho+\mu_m}(e_j)) \\ &= (p^\rho_{\rho-\mu_m}(e_j)^* p^\rho_{\rho-\mu_m}(e_i) + p^\rho_{\rho-\mu_m}(e_i)^* p^\rho_{\rho-\mu_m}(e_j)).\end{aligned}$$

This equation induces the identity (4.3). □

Remark 4. *The algebraic relation* (2.10) *induces other identities*

$$\hat{R}^{2q+1}_\rho = \sum_\lambda \{2\hat{m}(\lambda)^{2q+1} - \hat{m}(\lambda)^{2q} + \sum_{p=0}^{2q} \pi_\rho(\hat{c}_{2q-p}) \hat{m}(\lambda)^p\} (D^\rho_\lambda)^* D^\rho_\lambda$$

for $q = 0, 1, \cdots$ *. But, from the discussion below, these identities are linear dependent on* (4.1)*.*

We shall discuss linear independence of our Bochner identities (4.1). We assume that there are N irreducible components in $V_\rho \otimes \mathbb{R}^n$, and hence, N gradients $\{D^\rho_{\lambda_i}\}_{i=1}^N$ on $\Gamma(M, \mathbf{S}_\rho)$. Our aim is to prove that the identities (4.1) for $q = 1, \cdots, [N/2]$ are independent. In other words, if we define the vector $v(q)$ of the coefficients in (4.1) by

$$v(q) := (\sum_{p=0}^{2q-1} \pi_\rho(\hat{c}_{2q-1-p}) \hat{m}(\lambda_1)^p, \cdots, \sum_{p=0}^{2q-1} \pi_\rho(\hat{c}_{2q-1-p}) \hat{m}(\lambda_N)^p) \in \mathbb{R}^N,$$

then we would prove that $v(1), v(2), \cdots, v([N/2])$ are linear independent in $\mathbb{R}^N$.

For $q = 1, 2, \cdots$, we decompose $(v(1), v(2), \cdots, v(q))$ into the product of a $q \times 2q$ matrix $C(q)$ and a $2q \times N$ matrix $M(q)$ given by

$$C(q) := \begin{pmatrix} \pi_\rho(\hat{c}_1) & \pi_\rho(\hat{c}_0) & 0 & 0 & \cdots & 0 & 0 \\ \pi_\rho(\hat{c}_3) & \pi_\rho(\hat{c}_2) & \pi_\rho(\hat{c}_1) & \pi_\rho(\hat{c}_0) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \pi_\rho(\hat{c}_{2q-1}) & \pi_\rho(\hat{c}_{2q-2}) & \cdots & \cdots & \cdots & \pi_\rho(\hat{c}_1) & \pi_\rho(\hat{c}_0) \end{pmatrix},$$

$$M(q) := \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \hat{m}(\lambda_1) & \hat{m}(\lambda_2) & \hat{m}(\lambda_3) & \cdots & \hat{m}(\lambda_N) \\ \hat{m}(\lambda_1)^2 & \hat{m}(\lambda_2)^2 & \hat{m}(\lambda_3)^2 & \cdots & \hat{m}(\lambda_N)^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \hat{m}(\lambda_1)^{2q-1} & \hat{m}(\lambda_2)^{2q-1} & \hat{m}(\lambda_3)^{2q-1} & \cdots & \hat{m}(\lambda_N)^{2q-1} \end{pmatrix}.$$

Since the conformal weights are different from each other, the rank of the matrix $(v(1), v(2), \cdots, v([N/2])) = C([N/2])M([N/2])$ is $[N/2]$ except the exceptional case. For the exceptional case, the rank of $C([N/2])M([N/2])$ is $[N/2] - 1$. But, there is another identity (4.3) independent of (4.1).

Thus, if there are N gradients $\{D^\rho_{\lambda_i}\}_{i=1}^N$, then we have at least $[N/2]$ independent Bochner identities. In [1], by using spectral resolution on the standard sphere S^n, Branson proved that the number of independent identities is just $[N/2]$. Hence we conclude that

Corollary 3. *The identities* (4.1) *and* (4.3) *give all the Bochner identities for the gradients.*

References

[1] T. Branson, *Stein-Weiss operators and ellipticity,* J. Funct. Anal. **151**, (1997), 334–383.

[2] T. Branson, *Kato constants in Riemannian geometry,* Math. Res. Lett. **7** (2000), 245–261.

[3] T. Branson and O. Hijazi, *Vanishing theorems and eigenvalue estimates in Riemannian spin geometry,* Internat. J. Math. **8**, (1997), 921–934.

[4] T. Branson and O. Hijazi, *Improved forms of some vanishing theorems in Riemannian spin geometry,* Internat. J. Math., **11**, (2000), 291–304.

[5] J. Bureš, *The higher spin Dirac operators, in 'Differential geometry and applications',* Masaryk Univ., Brno, (1999), 319–334.

[6] D. Calderbank, P. Gauduchon, M. Herzlich, *Refined Kato inequalities and conformal weights in Riemannian geometry,* J. Funct. Anal. **173** (2000) 214–255

[7] H. D. Fegan, *Conformally invariant first order differential operators,* Quart. J. Math. Oxford, **27** (1976), 371–378.

[8] P. Gauduchon, *Structures de Weyl et theórèmes d'annulation sur une variété conforme autoduale,* Ann Scuola Norm. Sup. Pisa **18** (1991), 563–629.

[9] Y. Homma, *Clifford homomorphisms and higher spin Dirac operators,* preprint.

[10] Y. Homma, *Spherical harmonic polynomials for higher bundles, in 'Int. Conf. on Clifford Analysis, Its Appl. and Related Topics. Beijing',* Adv. Appl. Clifford Algebras **11** (S2) (2001), 117–126.

[11] Y. Homma, *The Bochner identities for the Kählerian gradients,* preprint.

[12] S. Okubo, *Casimir invariants and vector operators in simple and classical Lie algebras,* J. Math. Phys. **18**, (1977), 2382–2394

[13] D. P. Želobenko, *Compact Lie Groups and Their Representations,* Trans. Math. Monographs. vol **40**, A.M.S. 1973.

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